

“HCR’s Infinite-series”

Analysis of oblique frustum of a right circular cone

Mr Harish Chandra Rajpoot

Madan Mohan Malaviya University of Technology, Gorakhpur-273010 (UP) India

Abstract: Three series had been derived by the author, using double-integration in polar co-ordinates, binomial expansion and β & γ -functions for determining the volume, surface-area & perimeter of elliptical-section of oblique frustum of a right circular cone as there had not been any mathematical formula for determining the same due to some limitations. All these three series are in form of discrete summation of infinite terms which converge into finite values hence these were also named as HCR’s convergence series. These are extremely useful in case studies & practical computations.

Keywords: HCR’s Convergence-series, oblique frustum, binomial expansion, β & γ -functions

I. Introduction

When a right circular cone is thoroughly cut by a plane inclined at a certain angle with the axis of a right circular cone, an oblique frustum with elliptical section & apex point is generated. So far there are no mathematical formulae for determining the volume & surface area of oblique frustum & perimeter, major axis, minor axis & eccentricity of elliptical section. This article derives mathematical formulae for all the parameters.

II. Volume of oblique frustum

Let there be a right circular cone with apex angle 2α . Now, it is thoroughly cut by a plane (as shown by the extended line AB in fig 1.) inclined at an angle θ with axis OZ of the cone & lying at a normal distance h from the apex point O

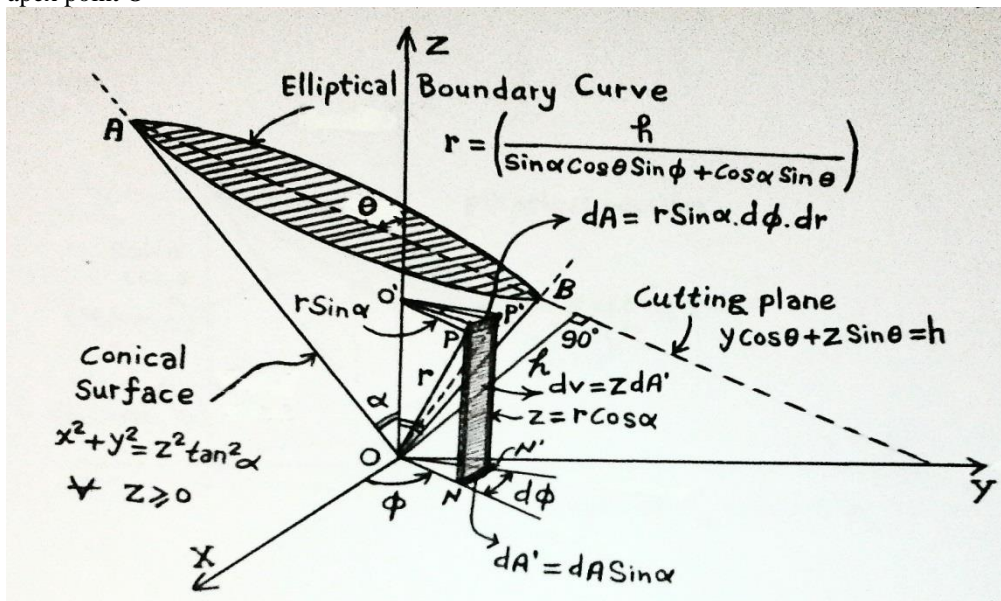


Fig 1: Oblique frustum of right cone with elliptical section

Now, consider any parametric point $P(r, \phi)$ on the conical surface

Equation of conical surface: in Cartesian co-ordinates

$$x^2 + y^2 = (z \tan\alpha)^2 = z^2 \tan^2\alpha \quad \forall z \geq 0$$

In Polar co-ordinates

$$x = (r \sin\alpha) \cos\phi$$

$$y = (r \sin\alpha) \sin\phi$$

$$z = r \cos\alpha$$

Equation of cutting plane: by intercept form

$$\frac{x}{\infty} + \frac{y}{h \sec \theta} + \frac{z}{h \operatorname{cosec} \theta} = 1 \Rightarrow y \cos \theta + z \sin \theta = h$$

Equation of elliptical boundary: Ellipse is the curve of intersection of conical surface & cutting plane. Hence on substituting the co-ordinates of parametric point $P(r, \varphi)$ in the equation of plane, we have the following equation in polar co-ordinates

$$\begin{aligned} \{(r \sin \alpha) \sin \varphi\} \cos \theta + \{r \cos \alpha\} \sin \theta &= h \\ \Rightarrow r &= \frac{h}{(\sin \alpha \cos \theta) \sin \varphi + \sin \theta \cos \alpha} \dots \dots \dots (I) \end{aligned}$$

Now, let's consider an imaginary cylindrical surface ABDC completely enclosing the frustum of solid cone (as shown in the fig 2. below). It is easier to calculate the volume of void space enclosed by the frustum of cone, cylindrical surface & XY-plane.

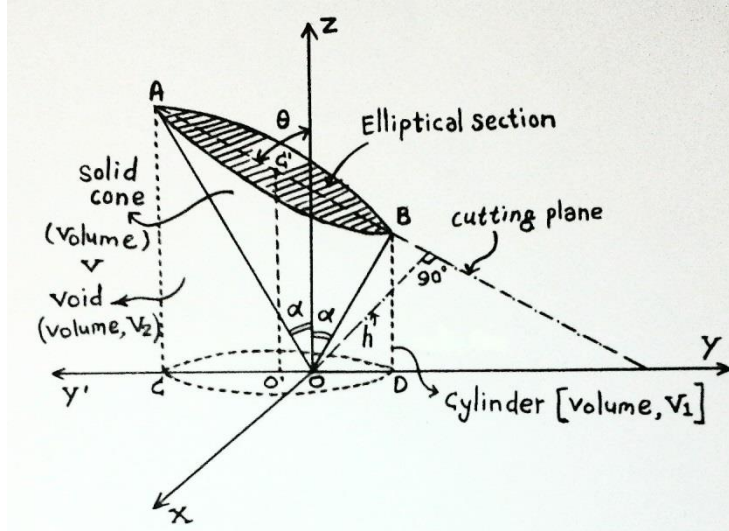


Fig 2: Imaginary cylindrical surface enclosing the frustum of cone

Let's consider an elementary surface area dA on the conical surface (see fig 1.)

$$dA = \{(r \sin \alpha) d\varphi\} dr = r \sin \alpha d\varphi dr$$

Taking the projection dA' of elementary area dA on XY-plane

$$\Rightarrow dA' = dA \sin \alpha = (r \sin \alpha d\varphi dr) \sin \alpha = r \sin^2 \alpha d\varphi dr$$

Now, volume of elementary vertical strip

$$dV = dA' \times z = dA' \times r \cos \alpha$$

$$\Rightarrow dV = (r \sin^2 \alpha d\varphi dr) r \cos \alpha = r^2 \sin^2 \alpha \cos \alpha d\varphi dr$$

Hence, using double integration with proper limits, total volume (V_2) of the enclosed void space

$$\begin{aligned} \Rightarrow V_2 &= \int dV = \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=r} r^2 \sin^2 \alpha \cos \alpha d\varphi dr \\ &= \sin^2 \alpha \cos \alpha \int_{\varphi=0}^{\varphi=2\pi} \left[\int_{r=0}^{r=r} r^2 dr \right] d\varphi \quad (\text{since, } \alpha = \text{constant}) \\ &= \sin^2 \alpha \cos \alpha \int_{\varphi=0}^{\varphi=2\pi} \left[\frac{r^3}{3} \right]_{r=0}^{r=r} d\varphi = \frac{\sin^2 \alpha \cos \alpha}{3} \int_{\varphi=0}^{\varphi=2\pi} r^3 d\varphi \end{aligned}$$

On substituting the value of r from eq(I), we have

$$\begin{aligned} V_2 &= \frac{\sin^2 \alpha \cos \alpha}{3} \int_{\varphi=0}^{\varphi=2\pi} \left(\frac{h}{(\sin \alpha \cos \theta) \sin \varphi + \sin \theta \cos \alpha} \right)^3 d\varphi \\ &= \frac{\sin^2 \alpha \cos \alpha}{3} \int_{\varphi=0}^{\varphi=2\pi} \frac{h^3}{(\sin \theta \cos \alpha)^3 \left(1 + \frac{\sin \alpha \cos \theta}{\sin \theta \cos \alpha} \sin \varphi \right)^3} d\varphi \end{aligned}$$

$$\begin{aligned}
&= \frac{h^3 \sin^2 \alpha \cos \alpha}{3(\sin \theta \cos \alpha)^3} \int_{\varphi=0}^{\varphi=2\pi} \frac{1}{\left(1 + \left(\frac{\tan \alpha}{\tan \theta}\right) \sin \varphi\right)^3} d\varphi \quad (h, \alpha \text{ \& } \theta \text{ are arbitrary constants}) \\
&= \frac{h^3 \tan^2 \alpha}{3 \sin^3 \theta} \int_{\varphi=0}^{\varphi=2\pi} \left(1 + \left(\frac{\tan \alpha}{\tan \theta}\right) \sin \varphi\right)^{-3} d\varphi
\end{aligned}$$

Since, the frustum has **finite elliptical section**, hence we have a condition

$$\begin{aligned}
0 < \alpha < \theta \leq \frac{\pi}{2} &\Rightarrow \frac{\tan \alpha}{\tan \theta} < 1 \\
\therefore \left| \left(\frac{\tan \alpha}{\tan \theta}\right) \sin \varphi \right| < 1 &\quad \forall 0 \leq \varphi \leq 2\pi
\end{aligned}$$

Now, using **Binomial Expansion**

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \dots \dots \infty \quad (\forall |x| < 1)$$

We have,

$$\Rightarrow V_2 = \frac{h^3 \tan^2 \alpha}{3 \sin^3 \theta} \int_{\varphi=0}^{\varphi=2\pi} (1 + C \sin \varphi)^{-3} d\varphi \quad \left(\text{let } \frac{\tan \alpha}{\tan \theta} = \text{constant} = C\right)$$

$$\begin{aligned}
&= \frac{h^3 \tan^2 \alpha}{3 \sin^3 \theta} \int_{\varphi=0}^{\varphi=2\pi} \left[1 - 3(C \sin \varphi) + \frac{3 \times 4}{2!} (C \sin \varphi)^2 - \frac{3 \times 4 \times 5}{3!} (C \sin \varphi)^3 + \frac{3 \times 4 \times 5 \times 6}{4!} (C \sin \varphi)^4 \right. \\
&\quad \left. - \frac{3 \times 4 \times 5 \times 6 \times 7}{5!} (C \sin \varphi)^5 + \dots \dots \dots \infty \right] d\varphi
\end{aligned}$$

$$\begin{aligned}
&= \frac{h^3 \tan^2 \alpha}{3 \sin^3 \theta} \left[\int_{\varphi=0}^{\varphi=2\pi} 1 \cdot d\varphi - \frac{1 \times 2 \times 3}{2} C \int_{\varphi=0}^{\varphi=2\pi} \sin \varphi d\varphi + \frac{1 \times 2 \times 3 \times 4}{2 \times 2!} C^2 \int_{\varphi=0}^{\varphi=2\pi} \sin^2 \varphi d\varphi \right. \\
&\quad - \frac{1 \times 2 \times 3 \times 4 \times 5}{2 \times 3!} C^3 \int_{\varphi=0}^{\varphi=2\pi} \sin^3 \varphi d\varphi + \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{2 \times 4!} C^4 \int_{\varphi=0}^{\varphi=2\pi} \sin^4 \varphi d\varphi \\
&\quad \left. - \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}{2 \times 5!} C^5 \int_{\varphi=0}^{\varphi=2\pi} \sin^5 \varphi d\varphi + \dots \dots \dots \infty \right]
\end{aligned}$$

(Multiplying & diving each term by 2 in above series)

$$\begin{aligned}
&= \frac{h^3 \tan^2 \alpha}{3 \sin^3 \theta} \left[2\pi - \frac{3!}{2} C \int_{\varphi=0}^{\varphi=2\pi} \sin \varphi d\varphi + \frac{4!}{2 \times 2!} C^2 \int_{\varphi=0}^{\varphi=2\pi} \sin^2 \varphi d\varphi - \frac{5!}{2 \times 3!} C^3 \int_{\varphi=0}^{\varphi=2\pi} \sin^3 \varphi d\varphi \right. \\
&\quad \left. + \frac{6!}{2 \times 4!} C^4 \int_{\varphi=0}^{\varphi=2\pi} \sin^4 \varphi d\varphi - \frac{7!}{2 \times 5!} C^5 \int_{\varphi=0}^{\varphi=2\pi} \sin^5 \varphi d\varphi + \dots \dots \dots \infty \right]
\end{aligned}$$

$$\text{since, } \int_0^{2a} f(x) dx = 0 \text{ if } f(2a - x) = -f(x)$$

$$= 2 \int_0^a f(x) dx \text{ if } f(2a - x) = f(x)$$

$$\therefore \int_{\varphi=0}^{\varphi=2\pi} \sin \varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi} \sin^3 \varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi} \sin^5 \varphi d\varphi = \dots \dots \dots = 0 \text{ \& }$$

$$\int_0^{2\pi} \sin^{2n} \varphi d\varphi = 2 \int_0^{\pi} \sin^{2n} \varphi d\varphi = 4 \int_0^{\frac{\pi}{2}} \sin^{2n} \varphi d\varphi \quad \forall n \in N$$

Now, we have

$$V_2 = \frac{h^3 \tan^2 \alpha}{3 \sin^3 \theta} \left[2\pi + 4 \frac{4!}{2 \times 2!} C^2 \int_0^{\frac{\pi}{2}} \sin^2 \varphi d\varphi + 4 \frac{6!}{2 \times 4!} C^4 \int_0^{\frac{\pi}{2}} \sin^4 \varphi d\varphi + 4 \frac{8!}{2 \times 6!} C^6 \int_0^{\frac{\pi}{2}} \sin^6 \varphi d\varphi \right.$$

$$\left. + \dots \dots \dots \infty \right]$$

$$= \frac{h^3 \tan^2 \alpha}{3 \sin^3 \theta} \left[2\pi + \frac{2 \times 4!}{2!} C^2 \int_0^{\frac{\pi}{2}} \sin^2 \varphi d\varphi + \frac{2 \times 6!}{4!} C^4 \int_0^{\frac{\pi}{2}} \sin^4 \varphi d\varphi + \frac{2 \times 8!}{6!} C^6 \int_0^{\frac{\pi}{2}} \sin^6 \varphi d\varphi + \dots \dots \dots \infty \right]$$

$$= \frac{2h^3 \tan^2 \alpha}{3 \sin^3 \theta} \left[\pi + \frac{4!}{2!} C^2 \int_0^{\frac{\pi}{2}} \sin^2 \phi d\phi + \frac{6!}{4!} C^4 \int_0^{\frac{\pi}{2}} \sin^4 \phi d\phi + \frac{8!}{6!} C^6 \int_0^{\frac{\pi}{2}} \sin^6 \phi d\phi + \dots \dots \dots \infty \right]$$

Since, we know from **β & γ -functions**

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)} \quad \& \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Hence, we have

$$\begin{aligned} V_2 &= \frac{2h^3 \tan^2 \alpha}{3 \sin^3 \theta} \left[\pi + \frac{4!}{2!} \left(\frac{\Gamma\left(\frac{2+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2+0+2}{2}\right)} \right) C^2 + \frac{6!}{4!} \left(\frac{\Gamma\left(\frac{4+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{4+0+2}{2}\right)} \right) C^4 \right. \\ &\quad \left. + \frac{8!}{6!} \left(\frac{\Gamma\left(\frac{6+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{6+0+2}{2}\right)} \right) C^6 + \dots \dots \dots \infty \right] \\ &= \frac{2h^3 \tan^2 \alpha}{3 \sin^3 \theta} \left[\pi + \frac{4!}{2!} \left(\frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)} \right) C^2 + \frac{6!}{4!} \left(\frac{\Gamma\left(\frac{5}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)} \right) C^4 + \frac{8!}{6!} \left(\frac{\Gamma\left(\frac{7}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(4)} \right) C^6 + \dots \dots \dots \infty \right] \\ &= \frac{2h^3 \tan^2 \alpha}{3 \sin^3 \theta} \left[\pi + \frac{4!}{2!} \left(\frac{\frac{1}{2} \sqrt{\pi} \times \sqrt{\pi}}{2 \times 1!} \right) C^2 + \frac{6!}{4!} \left(\frac{\frac{3}{2} \times \frac{1}{2} \sqrt{\pi} \times \sqrt{\pi}}{2 \times 2!} \right) C^4 + \frac{8!}{6!} \left(\frac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \sqrt{\pi} \times \sqrt{\pi}}{2 \times 3!} \right) C^6 \right. \\ &\quad \left. + \dots \dots \dots \infty \right] \\ &= \frac{2h^3 \tan^2 \alpha}{3 \sin^3 \theta} \left[\pi + \frac{4!}{2!} \times \frac{\pi}{2} \left(\frac{\frac{1}{2}}{1!} \right) C^2 + \frac{6!}{4!} \times \frac{\pi}{2} \left(\frac{\frac{3}{2} \times \frac{1}{2}}{2!} \right) C^4 + \frac{8!}{6!} \times \frac{\pi}{2} \left(\frac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}}{3!} \right) C^6 + \dots \dots \dots \infty \right] \\ &= \frac{2h^3 \tan^2 \alpha}{3 \sin^3 \theta} \times \pi \left[1 + \frac{4!}{2 \times 2!} \left(\frac{1}{2 \times 1!} \right) C^2 + \frac{6!}{2 \times 4!} \left(\frac{1 \times 3}{2^2 \times 2!} \right) C^4 + \frac{8!}{2 \times 6!} \left(\frac{1 \times 3 \times 5}{2^3 \times 3!} \right) C^6 + \dots \dots \dots \infty \right] \\ &= \frac{2\pi h^3 \tan^2 \alpha}{3 \sin^3 \theta} \left[1 + \frac{4!}{2 \times 2!} \left(\frac{1}{2 \times 1!} \right) C^2 + \frac{6!}{2 \times 4!} \left(\frac{1 \times 3}{2^2 \times 2!} \right) C^4 + \frac{8!}{2 \times 6!} \left(\frac{1 \times 3 \times 5}{2^3 \times 3!} \right) C^6 + \dots \dots \dots \infty \right] \\ &\Rightarrow V_2 = \left(\frac{2\pi h^3 \tan^2 \alpha}{3 \sin^3 \theta} \right) F_V \quad \dots \dots \dots (II) \end{aligned}$$

Where, $F_V \rightarrow$ is called Factor of volume or **HCR's F_V series**

$$\begin{aligned} F_V &= 1 + \frac{4!}{2 \times 2!} \left(\frac{1}{2 \times 1!} \right) C^2 + \frac{6!}{2 \times 4!} \left(\frac{1 \times 3}{2^2 \times 2!} \right) C^4 + \frac{8!}{2 \times 6!} \left(\frac{1 \times 3 \times 5}{2^3 \times 3!} \right) C^6 + \dots \dots \dots \infty \\ &= 1 + \frac{1}{2} \left[\frac{4!}{2!} \left(\frac{1}{2 \times 1!} \right) C^2 + \frac{6!}{4!} \left(\frac{1 \times 3}{2^2 \times 2!} \right) C^4 + \frac{8!}{6!} \left(\frac{1 \times 3 \times 5}{2^3 \times 3!} \right) C^6 + \frac{10!}{8!} \left(\frac{1 \times 3 \times 5 \times 7}{2^4 \times 4!} \right) C^8 + \dots \dots \dots \infty \right] \end{aligned}$$

In generalised form

$$\begin{aligned} F_V &= 1 + \frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{(2n+2)!}{(2n)!} \left\{ \frac{1 \times 3 \times 5 \times \dots \dots (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right] \\ &= 1 + \sum_{n=1}^{\infty} \left[\frac{(2n+2) \times (2n+1) \times (2n)!}{2 \times (2n)!} \left\{ \frac{1 \times 3 \times 5 \times \dots \dots (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right] \\ \Rightarrow F_V &= 1 + \sum_{n=1}^{\infty} \left[(n+1)(2n+1) \left\{ \frac{1 \times 3 \times 5 \times \dots \dots (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right] \quad \dots \dots \dots (III) \end{aligned}$$

Now, the **volume (V_1) of imaginary cylinder** enclosing the frustum of solid cone (see fig 2)

$$\begin{aligned} V_1 &= \frac{1}{2} [\pi(O'D)^2(AC + BD)] \\ &\Rightarrow O'D = BC' \sin \theta \end{aligned}$$

$$\text{where, } BC' = \text{semi major axis of ellipse} = \frac{h \sin \alpha \cos \alpha}{(\sin^2 \theta - \sin^2 \alpha)}$$

$$BD = \frac{h \cos \alpha}{\sin(\alpha + \theta)} \quad \&$$

$$AC = BD + AB \cos \theta = \frac{h \cos \alpha}{\sin(\alpha + \theta)} + \frac{2h \sin \alpha \cos \alpha \cos \theta}{(\sin^2 \theta - \sin^2 \alpha)}$$

Now, on setting the values, we get the volume of cylinder

$$\begin{aligned} V_1 &= \frac{1}{2} \left[\pi \left(\frac{h \sin \alpha \cos \alpha \sin \theta}{(\sin^2 \theta - \sin^2 \alpha)} \right)^2 \left(\frac{2h \cos \alpha}{\sin(\alpha + \theta)} + \frac{2h \sin \alpha \cos \alpha \cos \theta}{(\sin^2 \theta - \sin^2 \alpha)} \right) \right] \\ &= \frac{\pi h^3 \sin^2 \theta \sin^2 \alpha \cos^2 \alpha}{(\sin^2 \theta - \sin^2 \alpha)^2} \left(\frac{1}{\sin(\alpha + \theta)} + \frac{\sin \alpha \cos \theta}{(\sin^2 \theta - \sin^2 \alpha)} \right) \\ \Rightarrow V_1 &= \frac{\pi h^3 \sin^3 \theta \sin^2 \alpha \cos^4 \alpha}{(\sin^2 \theta - \sin^2 \alpha)^3} \end{aligned}$$

Hence, the volume (V) of the **oblique frustum** of right cone

$$V = \frac{\pi h^3 \sin^3 \theta \sin^2 \alpha \cos^4 \alpha}{(\sin^2 \theta - \sin^2 \alpha)^3} - \left(\frac{2\pi h^3 \tan^2 \alpha}{3 \sin^3 \theta} \right) F_V \quad \dots \dots \dots (IV)$$

Above is the required expression for calculating the volume of oblique frustum.

F_V series can be simplified as follows

$$F_V = 1 + 3C^2 + 5.625C^4 + 8.75C^6 + 12.3046875C^8 + 16.2421875C^{10} + 20.52832031C^{12} + 25.13671875C^{14} \dots \dots \dots \infty$$

III. Surface area of oblique frustum:

We know that the elementary area dA on the conical surface (see fig 1.)

$$dA = \{(r \sin \alpha) d\phi\} dr = r \sin \alpha d\phi dr$$

Hence, total surface area (S) of frustum of cone (using double integration with proper limits)

$$\begin{aligned} S &= \int dA = \int_{\phi=0}^{\phi=2\pi} \int_{r=0}^{r=r} r \sin \alpha d\phi dr \\ &= \sin \alpha \int_{\phi=0}^{\phi=2\pi} \left[\int_{r=0}^{r=r} r dr \right] d\phi \\ &= \sin \alpha \int_{\phi=0}^{\phi=2\pi} \left[\frac{r^2}{2} \right]_{r=0}^{r=r} d\phi = \frac{\sin \alpha}{2} \int_{\phi=0}^{\phi=2\pi} r^2 d\phi \end{aligned}$$

On substituting the value of r from eq(I), we have

$$\begin{aligned} S &= \frac{\sin \alpha}{2} \int_{\phi=0}^{\phi=2\pi} \left(\frac{h}{(\sin \alpha \cos \theta) \sin \phi + \sin \theta \cos \alpha} \right)^2 d\phi \\ &= \frac{\sin \alpha}{2} \int_{\phi=0}^{\phi=2\pi} \frac{h^2}{(\sin \theta \cos \alpha)^2 \left(1 + \left(\frac{\sin \alpha \cos \theta}{\sin \theta \cos \alpha} \right) \sin \phi \right)^2} d\phi \\ &= \frac{h^2 \sin \alpha}{2(\sin \theta \cos \alpha)^2} \int_{\phi=0}^{\phi=2\pi} \frac{1}{\left(1 + \left(\frac{\tan \alpha}{\tan \theta} \right) \sin \phi \right)^2} d\phi \quad (h, \alpha \text{ \& } \theta \text{ are arbitrary constants}) \\ &= \frac{h^2 \sec \alpha \tan \alpha}{2 \sin^2 \theta} \int_{\phi=0}^{\phi=2\pi} \left(1 + \left(\frac{\tan \alpha}{\tan \theta} \right) \sin \phi \right)^{-2} d\phi \end{aligned}$$

$$\text{since } \left| \left(\frac{\tan \alpha}{\tan \theta} \right) \sin \phi \right| < 1 \quad \forall 0 \leq \phi \leq 2\pi$$

Hence, using **binomial expansion** of $\left(1 + \left(\frac{\tan \alpha}{\tan \theta} \right) \sin \phi \right)^{-2}$

We have,

$$\Rightarrow S = \frac{h^2 \sec^2 \alpha}{2 \sin^2 \theta} \int_{\varphi=0}^{\varphi=2\pi} (1 + C \sin \varphi)^{-2} d\varphi \quad \left(\text{let } \frac{\tan \alpha}{\tan \theta} = \text{constant} = C \right)$$

$$S = \frac{h^2 \sec^2 \alpha}{2 \sin^2 \theta} \int_{\varphi=0}^{\varphi=2\pi} \left[1 - 2(C \sin \varphi) + \frac{2 \times 3}{2!} (C \sin \varphi)^2 - \frac{2 \times 3 \times 4}{3!} (C \sin \varphi)^3 + \frac{2 \times 3 \times 4 \times 5}{4!} (C \sin \varphi)^4 - \frac{2 \times 3 \times 4 \times 5 \times 6}{5!} (C \sin \varphi)^5 + \dots \dots \dots \infty \right] d\varphi$$

$$= \frac{h^2 \sec^2 \alpha}{2 \sin^2 \theta} \left[\int_{\varphi=0}^{\varphi=2\pi} 1 \cdot d\varphi - 1 \times 2C \int_{\varphi=0}^{\varphi=2\pi} \sin \varphi d\varphi + \frac{1 \times 2 \times 3}{2!} C^2 \int_{\varphi=0}^{\varphi=2\pi} \sin^2 \varphi d\varphi - \frac{1 \times 2 \times 3 \times 4}{3!} C^3 \int_{\varphi=0}^{\varphi=2\pi} \sin^3 \varphi d\varphi + \frac{1 \times 2 \times 3 \times 4 \times 5}{4!} C^4 \int_{\varphi=0}^{\varphi=2\pi} \sin^4 \varphi d\varphi - \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{5!} C^5 \int_{\varphi=0}^{\varphi=2\pi} \sin^5 \varphi d\varphi + \dots \dots \dots \infty \right]$$

$$= \frac{h^2 \sec^2 \alpha}{2 \sin^2 \theta} \left[2\pi - 2! C \int_{\varphi=0}^{\varphi=2\pi} \sin \varphi d\varphi + \frac{3!}{2!} C^2 \int_{\varphi=0}^{\varphi=2\pi} \sin^2 \varphi d\varphi - \frac{4!}{3!} C^3 \int_{\varphi=0}^{\varphi=2\pi} \sin^3 \varphi d\varphi + \frac{5!}{4!} C^4 \int_{\varphi=0}^{\varphi=2\pi} \sin^4 \varphi d\varphi - \frac{6!}{5!} C^5 \int_{\varphi=0}^{\varphi=2\pi} \sin^5 \varphi d\varphi + \dots \dots \dots \infty \right]$$

$$\text{since, } \int_0^{2a} f(x) dx = 0 \text{ if } f(2a - x) = -f(x)$$

$$= 2 \int_0^a f(x) dx \text{ if } f(2a - x) = f(x)$$

$$\therefore \int_{\varphi=0}^{\varphi=2\pi} \sin \varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi} \sin^3 \varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi} \sin^5 \varphi d\varphi = \dots \dots \dots = 0 \text{ \&}$$

$$\int_0^{2\pi} \sin^{2n} \varphi d\varphi = 2 \int_0^{\pi} \sin^{2n} \varphi d\varphi = 4 \int_0^{\frac{\pi}{2}} \sin^{2n} \varphi d\varphi \quad \forall n \in \mathbb{N}$$

Now, we have

$$S = \frac{h^2 \sec^2 \alpha}{2 \sin^2 \theta} \left[2\pi + \frac{4 \times 3!}{2!} C^2 \int_0^{\frac{\pi}{2}} \sin^2 \varphi d\varphi + \frac{4 \times 5!}{4!} C^4 \int_0^{\frac{\pi}{2}} \sin^4 \varphi d\varphi + \frac{4 \times 7!}{6!} C^6 \int_0^{\frac{\pi}{2}} \sin^6 \varphi d\varphi + \dots \dots \dots \infty \right]$$

$$= \frac{h^2 \sec^2 \alpha}{\sin^2 \theta} \left[\pi + \frac{2 \times 3!}{2!} C^2 \int_0^{\frac{\pi}{2}} \sin^2 \varphi d\varphi + \frac{2 \times 5!}{4!} C^4 \int_0^{\frac{\pi}{2}} \sin^4 \varphi d\varphi + \frac{2 \times 7!}{6!} C^6 \int_0^{\frac{\pi}{2}} \sin^6 \varphi d\varphi + \dots \dots \dots \infty \right]$$

Using β & γ -functions

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)} \text{ \& } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Hence, we have

$$S = \frac{h^2 \sec^2 \alpha}{\sin^2 \theta} \left[\pi + \frac{2 \times 3!}{2!} \left(\frac{\Gamma\left(\frac{2+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2+0+2}{2}\right)} \right) C^2 + \frac{2 \times 5!}{4!} \left(\frac{\Gamma\left(\frac{4+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{4+0+2}{2}\right)} \right) C^4 + \frac{2 \times 7!}{6!} \left(\frac{\Gamma\left(\frac{6+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{6+0+2}{2}\right)} \right) C^6 + \dots \dots \dots \infty \right]$$

$$\begin{aligned}
&= \frac{h^2 \sec \alpha \tan \alpha}{\sin^2 \theta} \left[\pi + \frac{2 \times 3!}{2!} \left(\frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)} \right) C^2 + \frac{2 \times 5!}{4!} \left(\frac{\Gamma\left(\frac{5}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)} \right) C^4 + \frac{2 \times 7!}{6!} \left(\frac{\Gamma\left(\frac{7}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(4)} \right) C^6 \right. \\
&\quad \left. + \dots \dots \dots \infty \right] \\
&= \frac{h^2 \sec \alpha \tan \alpha}{\sin^2 \theta} \left[\pi + \frac{2 \times 3!}{2!} \left(\frac{\frac{1}{2} \sqrt{\pi} \times \sqrt{\pi}}{2 \times 1!} \right) C^2 + \frac{2 \times 5!}{4!} \left(\frac{\frac{3}{2} \times \frac{1}{2} \sqrt{\pi} \times \sqrt{\pi}}{2 \times 2!} \right) C^4 \right. \\
&\quad \left. + \frac{2 \times 7!}{6!} \left(\frac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \sqrt{\pi} \times \sqrt{\pi}}{2 \times 3!} \right) C^6 + \dots \dots \dots \infty \right] \\
&= \frac{h^2 \sec \alpha \tan \alpha}{\sin^2 \theta} \left[\pi + \frac{2 \times 3!}{2!} \times \frac{\pi}{2} \left(\frac{1}{2!} \right) C^2 + \frac{2 \times 5!}{4!} \times \frac{\pi}{2} \left(\frac{3 \times 1}{2!} \right) C^4 + \frac{27 \times 7!}{6!} \times \frac{\pi}{2} \left(\frac{5 \times 3 \times 1}{3!} \right) C^6 \right. \\
&\quad \left. + \dots \dots \dots \infty \right] \\
&= \frac{h^2 \sec \alpha \tan \alpha}{\sin^2 \theta} \times \pi \left[1 + \frac{3!}{2!} \left(\frac{1}{2 \times 1!} \right) C^2 + \frac{5!}{4!} \left(\frac{1 \times 3}{2^2 \times 2!} \right) C^4 + \frac{7!}{6!} \left(\frac{1 \times 3 \times 5}{2^3 \times 3!} \right) C^6 + \dots \dots \dots \infty \right] \\
&= \frac{\pi h^2 \sec \alpha \tan \alpha}{\sin^2 \theta} \left[1 + \frac{3!}{2!} \left(\frac{1}{2 \times 1!} \right) C^2 + \frac{5!}{4!} \left(\frac{1 \times 3}{2^2 \times 2!} \right) C^4 + \frac{7!}{6!} \left(\frac{1 \times 3 \times 5}{2^3 \times 3!} \right) C^6 + \dots \dots \dots \infty \right] \\
&\Rightarrow S = \left(\frac{\pi h^2 \sec \alpha \tan \alpha}{\sin^2 \theta} \right) F_S \quad \dots \dots \dots (V)
\end{aligned}$$

Above is the required expression for calculating the surface area of oblique frustum.

Where, $F_S \rightarrow$ is called Factor of surface area or **HCR's F_S series**

$$F_S = 1 + \frac{3!}{2!} \left(\frac{1}{2 \times 1!} \right) C^2 + \frac{5!}{4!} \left(\frac{1 \times 3}{2^2 \times 2!} \right) C^4 + \frac{7!}{6!} \left(\frac{1 \times 3 \times 5}{2^3 \times 3!} \right) C^6 + \dots \dots \dots \infty$$

In generalised form

$$\begin{aligned}
F_S &= 1 + \sum_{n=1}^{\infty} \left[\frac{(2n+1)!}{(2n)!} \left\{ \frac{1 \times 3 \times 5 \times \dots \dots (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right] \\
&= 1 + \sum_{n=1}^{\infty} \left[\frac{(2n+1) \times (2n)!}{(2n)!} \left\{ \frac{1 \times 3 \times 5 \times \dots \dots (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right] \\
\Rightarrow F_S &= 1 + \sum_{n=1}^{\infty} \left[(2n+1) \left\{ \frac{1 \times 3 \times 5 \times \dots \dots (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right] \quad \dots \dots \dots (VI)
\end{aligned}$$

F_S series can be simplified as follows

$$F_S = 1 + 1.5C^2 + 1.875C^4 + 2.1875C^6 + 2.4609375C^8 + 2.70703125C^{10} + 2.932617188C^{12} + 3.142089844C^{14} + \dots \dots \dots \infty$$

IV. Perimeter of elliptical section of oblique frustum:

Now, consider any parametric point $P(r, \varphi)$ on the periphery of elliptical section (see fig 3. below)

Now, the elementary perimeter $PP' = dP = O'P \times \angle PO'P'$

$$dP = (r \sin \alpha) \times \left(\frac{d\varphi}{\sin \theta} \right) = \frac{r \sin \alpha d\varphi}{\sin \theta}$$

Hence, the total perimeter (P) of the elliptical section (using integration with proper limits)

$$P = \int dP = \int_{\varphi=0}^{\varphi=2\pi} \frac{r \sin \alpha d\varphi}{\sin \theta}$$

$$= \frac{\sin\alpha}{\sin\theta} \int_{\varphi=0}^{\varphi=2\pi} r d\varphi$$

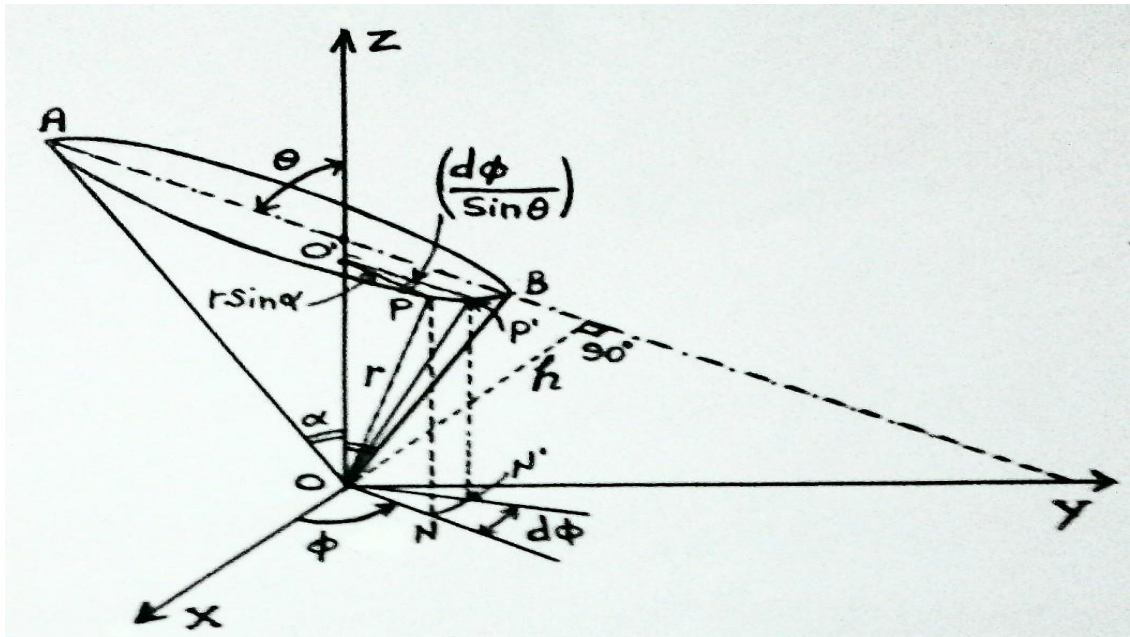


Fig 3: Perimeter of elliptical section of oblique frustum

On substituting the value of r from eq(I), we have

$$\begin{aligned} P &= \frac{\sin\alpha}{\sin\theta} \int_{\varphi=0}^{\varphi=2\pi} \left(\frac{h}{(\sin\alpha\cos\theta)\sin\varphi + \sin\theta\cos\alpha} \right) d\varphi \\ &= \frac{\sin\alpha}{\sin\theta} \int_{\varphi=0}^{\varphi=2\pi} \frac{h}{(\sin\theta\cos\alpha) \left(1 + \left(\frac{\sin\alpha\cos\theta}{\sin\theta\cos\alpha} \right) \sin\varphi \right)} d\varphi \\ &= \frac{h\sin\alpha}{\sin\theta(\sin\theta\cos\alpha)} \int_{\varphi=0}^{\varphi=2\pi} \frac{1}{\left(1 + \left(\frac{\tan\alpha}{\tan\theta} \right) \sin\varphi \right)} d\varphi \quad (h, \alpha \text{ \& } \theta \text{ are arbitrary constants}) \\ &= \frac{h\tan\alpha}{\sin^2\theta} \int_{\varphi=0}^{\varphi=2\pi} \left(1 + \left(\frac{\tan\alpha}{\tan\theta} \right) \sin\varphi \right)^{-1} d\varphi \end{aligned}$$

$$\text{since } \left| \left(\frac{\tan\alpha}{\tan\theta} \right) \sin\varphi \right| < 1 \quad \forall 0 \leq \varphi \leq 2\pi$$

Hence, using **binomial expansion** of $\left(1 + \left(\frac{\tan\alpha}{\tan\theta} \right) \sin\varphi \right)^{-1}$ as follows
 $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \dots \dots \dots \infty \quad (\forall |x| < 1)$

We have,

$$\Rightarrow P = \frac{h\tan\alpha}{\sin^2\theta} \int_{\varphi=0}^{\varphi=2\pi} (1 + C\sin\varphi)^{-1} d\varphi \quad \left(\text{let } \frac{\tan\alpha}{\tan\theta} = \text{constant} = C \right)$$

$$P = \frac{h\tan\alpha}{\sin^2\theta} \int_{\varphi=0}^{\varphi=2\pi} [1 - (C\sin\varphi) + (C\sin\varphi)^2 - (C\sin\varphi)^3 + (C\sin\varphi)^4 - (C\sin\varphi)^5 + \dots \dots \dots \infty] d\varphi$$

$$= \frac{htana}{\sin^2\theta} \left[\int_{\varphi=0}^{\varphi=2\pi} 1. d\varphi - C \int_{\varphi=0}^{\varphi=2\pi} \sin\varphi d\varphi + C^2 \int_{\varphi=0}^{\varphi=2\pi} \sin^2\varphi d\varphi - C^3 \int_{\varphi=0}^{\varphi=2\pi} \sin^3\varphi d\varphi + C^4 \int_{\varphi=0}^{\varphi=2\pi} \sin^4\varphi d\varphi - C^5 \int_{\varphi=0}^{\varphi=2\pi} \sin^5\varphi d\varphi + C^6 \int_{\varphi=0}^{\varphi=2\pi} \sin^6\varphi d\varphi - \dots \dots \dots \infty \right]$$

$$= \frac{htana}{\sin^2\theta} \left[2\pi - C \int_{\varphi=0}^{\varphi=2\pi} \sin\varphi d\varphi + C^2 \int_{\varphi=0}^{\varphi=2\pi} \sin^2\varphi d\varphi - C^3 \int_{\varphi=0}^{\varphi=2\pi} \sin^3\varphi d\varphi + C^4 \int_{\varphi=0}^{\varphi=2\pi} \sin^4\varphi d\varphi - C^5 \int_{\varphi=0}^{\varphi=2\pi} \sin^5\varphi d\varphi + C^6 \int_{\varphi=0}^{\varphi=2\pi} \sin^6\varphi d\varphi - \dots \dots \dots \infty \right]$$

using property of limit $\int_0^{2a} f(x) dx = 0$ if $f(2a - x) = -f(x)$

$$= 2 \int_0^a f(x) dx \text{ if } f(2a - x) = f(x)$$

$$\therefore \int_{\varphi=0}^{\varphi=2\pi} \sin\varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi} \sin^3\varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi} \sin^5\varphi d\varphi = \dots \dots \dots = 0 \text{ \&}$$

$$\int_0^{2\pi} \sin^{2n}\varphi d\varphi = 2 \int_0^{\pi} \sin^{2n}\varphi d\varphi = 4 \int_0^{\frac{\pi}{2}} \sin^{2n}\varphi d\varphi \quad \forall n \in N$$

Now, we have

$$P = \frac{htana}{\sin^2\theta} \left[2\pi + 4C^2 \int_0^{\frac{\pi}{2}} \sin^2\varphi d\varphi + 4C^4 \int_0^{\frac{\pi}{2}} \sin^4\varphi d\varphi + 4C^6 \int_0^{\frac{\pi}{2}} \sin^6\varphi d\varphi + \dots \dots \dots \infty \right]$$

$$= \frac{2htana}{\sin^2\theta} \left[\pi + 2C^2 \int_0^{\frac{\pi}{2}} \sin^2\varphi d\varphi + 2C^4 \int_0^{\frac{\pi}{2}} \sin^4\varphi d\varphi + 2C^6 \int_0^{\frac{\pi}{2}} \sin^6\varphi d\varphi + \dots \dots \dots \infty \right]$$

Now, using **β & γ -functions**

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)} \text{ \& } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Hence, we have

$$P = \frac{2htana}{\sin^2\theta} \left[\pi + 2 \left(\frac{\Gamma\left(\frac{2+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2+0+2}{2}\right)} \right) C^2 + 2 \left(\frac{\Gamma\left(\frac{4+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{4+0+2}{2}\right)} \right) C^4 + 2 \left(\frac{\Gamma\left(\frac{6+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{6+0+2}{2}\right)} \right) C^6 + \dots \dots \dots \infty \right]$$

$$= \frac{2htana}{\sin^2\theta} \left[\pi + 2 \left(\frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)} \right) C^2 + 2 \left(\frac{\Gamma\left(\frac{5}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)} \right) C^4 + 2 \left(\frac{\Gamma\left(\frac{7}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(4)} \right) C^6 + \dots \dots \dots \infty \right]$$

$$= \frac{2htana}{\sin^2\theta} \left[\pi + 2 \left(\frac{\frac{1}{2}\sqrt{\pi} \times \sqrt{\pi}}{2 \times 1!} \right) C^2 + 2 \left(\frac{\frac{3}{2} \times \frac{1}{2}\sqrt{\pi} \times \sqrt{\pi}}{2 \times 2!} \right) C^4 + 2 \left(\frac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi} \times \sqrt{\pi}}{2 \times 3!} \right) C^6 + \dots \dots \dots \infty \right]$$

$$= \frac{2htana}{\sin^2\theta} \left[\pi + 2 \times \frac{\pi}{2} \left(\frac{1}{1!} \right) C^2 + 2 \times \frac{\pi}{2} \left(\frac{\frac{3}{2} \times \frac{1}{2}}{2!} \right) C^4 + 2 \times \frac{\pi}{2} \left(\frac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}}{3!} \right) C^6 + \dots \dots \dots \infty \right]$$

$$= \frac{2htana}{\sin^2\theta} \times \pi \left[1 + \left(\frac{1}{2 \times 1!} \right) C^2 + \left(\frac{1 \times 3}{2^2 \times 2!} \right) C^4 + \left(\frac{1 \times 3 \times 5}{2^3 \times 3!} \right) C^6 + \dots \dots \dots \infty \right]$$

$$= \frac{2\pi htana}{\sin^2\theta} \left[1 + \left(\frac{1}{2 \times 1!} \right) C^2 + \left(\frac{1 \times 3}{2^2 \times 2!} \right) C^4 + \left(\frac{1 \times 3 \times 5}{2^3 \times 3!} \right) C^6 + \dots \dots \dots \infty \right]$$

$$\Rightarrow P = \left(\frac{2\pi h \tan \alpha}{\sin^2 \theta} \right) F_P \quad \dots \dots \dots (VII)$$

Above is the required expression for calculating the perimeter of elliptical section of oblique frustum. Where, $F_P \rightarrow$ is called Factor of surface area or **HCR's F_P series**

$$F_P = 1 + \left(\frac{1}{2 \times 1!} \right) C^2 + \left(\frac{1 \times 3}{2^2 \times 2!} \right) C^4 + \left(\frac{1 \times 3 \times 5}{2^3 \times 3!} \right) C^6 + \dots \dots \dots \infty$$

In generalised form

$$F_P = 1 + \sum_{n=1}^{\infty} \left[\left\{ \frac{1 \times 3 \times 5 \times \dots \dots \dots (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right]$$

$$\Rightarrow F_P = 1 + \sum_{n=1}^{\infty} \left[\left\{ \frac{1 \times 3 \times 5 \times \dots \dots \dots (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right] \quad \dots \dots \dots (VIII)$$

Above series can be simplified as follows

$$F_P = 1 + 0.5C^2 + 0.375C^4 + 0.3125C^6 + 0.2734375C^8 + 0.24609375C^{10} + 0.225585937C^{12} + 0.209472656C^{14} + \dots \dots \dots \infty$$

IV. Conclusion:

Thus, all the results can concluded from equations III, IV, V, VI, VII & VIII as follows

Let a right circular cone with apex angle 2α be thoroughly cut by a plane inclined at an angle θ with axis OO' of the cone & lying at a normal distance h from the apex point O (as shown in the fig 4 below)

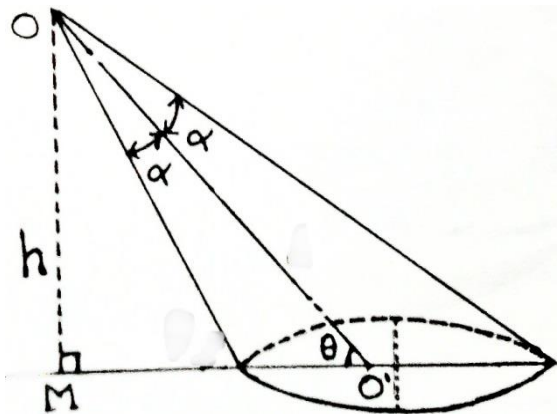


Fig. 4

Now, consider the oblique frustum with elliptical section & apex point 'O' (see above fig 4) The following parameters can be determined as tabulated below

<p>Volume (V) of frustum</p>	$V = \frac{\pi h^3 \sin^3 \theta \sin^2 \alpha \cos^4 \alpha}{(\sin^2 \theta - \sin^2 \alpha)^3} - \left(\frac{2\pi h^3 \tan^2 \alpha}{3 \sin^3 \theta} \right) F_V$ <p>where</p> $F_V = 1 + \sum_{n=1}^{\infty} \left[(n+1)(2n+1) \left\{ \frac{1 \times 3 \times 5 \times \dots \dots \dots (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right]$ $= 1 + 3C^2 + 5.625C^4 + 8.75C^6 + 12.3046875C^8 + 16.2421875C^{10} + 20.52832031C^{12} + 25.13671875C^{14} + \dots \dots \dots \infty$
<p>Surface Area (S) of frustum</p>	$S = \left(\frac{\pi h^2 \sec \alpha \tan \alpha}{\sin^2 \theta} \right) F_S$ <p>where</p>

	$F_S = 1 + \sum_{n=1}^{\infty} \left[(2n+1) \left\{ \frac{1 \times 3 \times 5 \times \dots \times (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right]$ $= 1 + 1.5C^2 + 1.875C^4 + 2.1875C^6 + 2.4609375C^8 + 2.70703125C^{10}$ $+ 2.932617188C^{12} + 3.142089844C^{14} + \dots \dots \dots \infty$
Perimeter (P) of Elliptical section	<p>where</p> $P = \left(\frac{2\pi h \tan \alpha}{\sin^2 \theta} \right) F_P$ $F_P = 1 + \sum_{n=1}^{\infty} \left[\left\{ \frac{1 \times 3 \times 5 \times \dots \times (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right]$ $= 1 + 0.5C^2 + 0.375C^4 + 0.3125C^6 + 0.2734375C^8 + 0.24609375C^{10}$ $+ 0.225585937C^{12} + 0.209472656C^{14} + \dots \dots \dots \infty$
Major Axis (2a)	$2a = \frac{h \sin 2\alpha}{(\sin^2 \theta - \sin^2 \alpha)}$
Minor Axis (2b)	$2b = \frac{2h \sin \alpha}{\sqrt{(\sin^2 \theta - \sin^2 \alpha)}}$
Eccentricity (e)	$e = \frac{\cos \theta}{\cos \alpha}$
Conditions for known quantities h, α & θ	<p>where</p> $0 < \alpha < \theta \leq \frac{\pi}{2} \quad \& \quad C = \left(\frac{\tan \alpha}{\tan \theta} \right) < 1$ <p>$\alpha \rightarrow$ semi apex angle of right circular cone $\theta \rightarrow$ angle between cutting plane & axis of cone $h \rightarrow$ normal distance of cutting plane from the apex point of cone</p>

Important deductions

- All the series converge to a finite value for the given values of α & θ . Although, the order of convergence rates of F_V, F_S & F_P is as follows $F_V \leq F_S \leq F_P$ & $F_V = F_S = F_P = 1$ for $\theta = 90^\circ$
- It is obvious that the rate of convergence of F_V & F_S depends on the values of α & θ
 - If $\theta - \alpha > 5^\circ \Rightarrow F_V$ & F_S converge to finite values with less number of terms i.e. terms of higher power can be neglected with insignificant errors in the results
 - If $\theta - \alpha \leq 5^\circ \Rightarrow F_V$ & F_S converge to finite values with more number of terms i.e. neglecting the terms of higher power may cause some error. In this case, the values of F_V & F_S must be determined by taking more number of terms until higher power terms become significantly negligible.
- If a right circular cone with apex angle 2α is thoroughly cut by a plane normal to the axis of the cone & lying at a normal distance h from the apex point then by setting $\theta = 90^\circ$ in the above results, we have the following results for the frustum generated

a. Volume of frustum

$$V = \frac{\pi h^3 \sin^3 90^\circ \sin^2 \alpha \cos^4 \alpha}{(\sin^2 90^\circ - \sin^2 \alpha)^3} - \left(\frac{2\pi h^3 \tan^2 \alpha}{3 \sin^3 90^\circ} \right) F_V$$

$$\begin{aligned}
&= \frac{\pi h^3 \sin^2 \alpha \cos^4 \alpha}{(1 - \sin^2 \alpha)^3} - \left(\frac{2\pi h^3 \tan^2 \alpha}{3} \right) \times 1 \quad (\text{since, } F_V = 1 \text{ for } \theta = 90^\circ) \\
&= \frac{\pi h^3 \sin^2 \alpha \cos^4 \alpha}{\cos^6 \alpha} - \frac{2\pi h^3 \tan^2 \alpha}{3} = \pi h^3 \tan^2 \alpha - \frac{2\pi h^3 \tan^2 \alpha}{3} \\
&= \frac{1}{3} \pi h^3 \tan^2 \alpha = \frac{1}{3} \pi (h \tan \alpha)^2 \times h = \frac{1}{3} \pi (\text{radius})^2 \times (\text{normal height}) \\
&= \text{volume of a right circular cone with apex angle } 2\alpha \text{ \& normal height } h
\end{aligned}$$

b. Surface area of frustum

$$\begin{aligned}
S &= \left(\frac{\pi h^2 \sec \alpha \tan \alpha}{\sin^2 90^\circ} \right) F_S \\
&= \left(\frac{\pi h^2 \sec \alpha \tan \alpha}{1} \right) \times 1 \quad (\text{since, } F_S = 1 \text{ for } \theta = 90^\circ) \\
&= \pi h^2 \sec \alpha \tan \alpha = \pi (h \tan \alpha) (h \sec \alpha) = \pi \times (\text{radius}) \times (\text{slant height}) \\
&= \text{surface area of right circular cone with apex angle } 2\alpha \text{ \& normal height } h
\end{aligned}$$

c. Perimeter of section generated

$$\begin{aligned}
P &= \left(\frac{2\pi h \tan \alpha}{\sin^2 90^\circ} \right) F_P \\
&= \left(\frac{2\pi h \tan \alpha}{1} \right) \times 1 = 2\pi (h \tan \alpha) = 2\pi \times (\text{radius}) \quad (\text{since, } F_P = 1 \text{ for } \theta = 90^\circ) \\
&= \text{periphery of base of right circular cone with apex angle } 2\alpha \text{ \& normal height } h
\end{aligned}$$

d. Major axis of generated section

$$\begin{aligned}
2a &= \frac{h \sin 2\alpha}{(\sin^2 90^\circ - \sin^2 \alpha)} \\
&= \frac{h \sin 2\alpha}{(1 - \sin^2 \alpha)} = \frac{2h \sin \alpha \cos \alpha}{\cos^2 \alpha} = 2(h \tan \alpha) = 2 \times (\text{radius}) \\
&= \text{diameter of base of right circular cone with apex angle } 2\alpha \text{ \& normal height } h
\end{aligned}$$

e. Minor axis of generated section

$$\begin{aligned}
2b &= \frac{2h \sin \alpha}{\sqrt{(\sin^2 90^\circ - \sin^2 \alpha)}} \\
&= \frac{2h \sin \alpha}{\sqrt{(1 - \sin^2 \alpha)}} = \frac{2h \sin \alpha}{\cos \alpha} = 2(h \tan \alpha) = 2 \times (\text{radius}) \\
&= \text{diameter of base of right circular cone with apex angle } 2\alpha \text{ \& normal height } h
\end{aligned}$$

f. Eccentricity of generated section

$$e = \frac{\cos 90^\circ}{\cos \alpha} = 0 = \text{eccentricity of a circle}$$

It is clear from the above results that when a right circular cone is thoroughly cut with a plane normal to the axis of cone generates a frustum which is itself a right cone with circular section. Hence, all the mathematical results derived by the author are verified.

Thus, the above results are very useful for analysis of oblique frustum in determining the volume, surface area & perimeter of elliptical section. All the derived results can be easily verified by the experimental results. These results can be used in practical applications, case studies & other academic purposes.

Illustrative Example

Consider a right circular cone with apex angle $2\alpha = 60^\circ$, let it be thoroughly cut by a smooth plane inclined at an angle $\theta = 70^\circ$ with the axis & lying at a normal distance $h = 20\text{cm}$. from the apex point.

In this case, $\theta - \alpha = 70^\circ - 30^\circ = 40^\circ \gg 5^\circ$ hence the rate of convergence of F_V, F_S & F_P series is much higher i.e. terms of higher power can be neglected with no significant error in the results

Let's first calculate the values of F_V, F_S & F_P for $\alpha = 30^\circ$ & $\theta = 70^\circ$ as follows

$$F_V = 1 + 3C^2 + 5.625C^4 + 8.75C^6 + 12.3046875C^8 + 16.2421875C^{10} + 20.52832031C^{12} + 25.13671875C^{14} + \dots \dots \dots \infty$$

On setting the value of constant, $C = \left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)$, we have

$$F_V = 1 + 3\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^2 + 5.625\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^4 + 8.75\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^6 + 12.3046875\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^8 + 16.2421875\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^{10} + 20.52832031\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^{12} + 25.13671875\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^{14} + \dots \dots \dots \infty$$

$$= 1 + 0.132474331 + 0.010968405 + 0.00075342440828 + 0.00004678563359 + 0.000002727074033 + 0.0000001522005794 + 0.00000008229661223 + \dots \dots \dots \infty$$

$$\Rightarrow F_V \approx \mathbf{1.144245834}$$

similarly, $F_S = 1 + 1.5\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^2 + 1.875\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^4 + 2.1875\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^6 + 2.4609375\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^8 + 2.70703125\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^{10} + 2.932617188\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^{12} + 3.142089844\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^{14} + \dots \dots \dots \infty$

$$= 1 + 0.066237165 + 0.003656135102 + 0.0001883560207 + 0.000009357126717 + 0.0000004545123389 + 0.00000002174293992 + 0.000000001028707653 + \dots \dots \dots \infty$$

$$\Rightarrow F_S \approx \mathbf{1.070091491}$$

similarly, $F_P = 1 + 0.5\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^2 + 0.375\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^4 + 0.3125\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^6 + 0.2734375\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^8 + 0.24609375\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^{10} + 0.225585937\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^{12} + 0.209472656\left(\frac{\tan 30^\circ}{\tan 70^\circ}\right)^{14} + \dots \dots \dots \infty$

$$= 1 + 0.022079055 + 0.0007312270203 + 0.00002690800296 + 0.000001046928806 + 0.00000004131930353 + 0.000000001672533836 + 0.00000000006858051011 + \dots \dots \dots \infty$$

$$\Rightarrow F_P \approx \mathbf{1.02283828}$$

Now, on setting the corresponding values of $\alpha, \theta, h, F_V, F_S$ & F_P we get the following results

a. Volume of frustum

$$V = \frac{\pi h^3 \sin^3 \theta \sin^2 \alpha \cos^4 \alpha}{(\sin^2 \theta - \sin^2 \alpha)^3} - \left(\frac{2\pi h^3 \tan^2 \alpha}{3 \sin^3 \theta}\right) F_V$$

$$= \frac{\pi(20)^3 \sin^3 70^\circ \sin^2 30^\circ \cos^4 30^\circ}{(\sin^2 70^\circ - \sin^2 30^\circ)^3} - \left(\frac{2\pi(20)^3 \tan^2 30^\circ}{3 \sin^3 70^\circ}\right) \times 1.144245834$$

$$= 3859.458211 \text{ cm}^3$$

b. Surface area of frustum

$$\begin{aligned} S &= \left(\frac{\pi h^2 \sec \alpha \tan \alpha}{\sin^2 \theta} \right) F_S \\ &= \left(\frac{\pi (20)^2 \sec 30^\circ \tan 30^\circ}{\sin^2 70^\circ} \right) \times 1.070091491 = 1015.238041 \text{ cm}^2 \end{aligned}$$

c. Perimeter of section generated

$$\begin{aligned} P &= \left(\frac{2\pi h \tan \alpha}{\sin^2 \theta} \right) F_P \\ &= \left(\frac{2\pi (20) \tan 30^\circ}{\sin^2 70^\circ} \right) \times 1.02283828 = 84.03971618 \text{ cm} \end{aligned}$$

d. Major axis of generated section

$$\begin{aligned} 2a &= \frac{h \sin 2\alpha}{(\sin^2 90^\circ - \sin^2 \alpha)} \\ &= \frac{(20) \sin(2 \times 30^\circ)}{(\sin^2 70^\circ - \sin^2 30^\circ)} = 27.36161147 \text{ cm} \end{aligned}$$

e. Minor axis of generated section

$$\begin{aligned} 2b &= \frac{2h \sin \alpha}{\sqrt{(\sin^2 \theta - \sin^2 \alpha)}} \\ &= \frac{2(20) \sin 30^\circ}{\sqrt{(\sin^2 70^\circ - \sin^2 30^\circ)}} = 25.13740937 \text{ cm} \end{aligned}$$

f. Eccentricity of generated section

$$e = \frac{\cos 70^\circ}{\cos 30^\circ} = 0.394930843$$

Thus, all the mathematical results obtained above can be verified by the experimental results. The symbols & names used above are arbitrary given by the author Mr Harish Chandra Rajpoot (B Tech, Mech. Engg.).

Acknowledgements:

Author is extremely thankful to all his teacher & professors who inspired & guided him for the research work in the field of Applied Mathematics. He also thanks to his parents & his elder brother who have been showering their blessing on him. Finally, he thanks to Almighty God who kept him physically fit during the completion of his research work.

Reference:

- [1]. The work of the author is original. It is an outcome of deep studies & experiments carried out by the author on the analysis of oblique frustum of right circular cone.
- [2]. Some results are directly taken from "Advanced Geometry", a book of research articles, by the author published with Notion Press, India (www.notionpress.com)
- [3]. Previous paper entitled "HCR's Rank Formula" was subjected to the peer review by Dr K. Srinivasa Rao, Senior professor, Distinguished Professor (DST-Ramanujan Prof.), director of Ramanujan Academy of Maths Talent, Luz Church Road, Chennai, India.