

ADVANCED GEOMETRY BY HARISH CHANDRA RAJPOOT

Hand Book

Formula Kit

COMPOSED & COMPILED

BY

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All the derivations & numerical data of this book, calculated by the author Mr H.C. Rajpoot, are mathematically correct to the best of his knowledge & experience.

Author's Sign

Harishchandra

CONTENTS

Topics	Page No
❖ Fundamental & Integral Theorems of Solid Angle	3
➤ Basic Definition & Formulation	
❖ Solid Angle subtended by Plane Figures	4
➤ Right Triangular Plane	
➤ Rectangular Plane	
➤ Square Plane	
➤ Regular Polygonal Plane	
❖ Approximate Analysis of Solid Angle	8
➤ Solid Angle subtended by a Symmetrical Plane	
➤ Elliptical Plane	
➤ Circular Plane	
➤ Regular Polygonal Plane	
➤ Rectangular Plane	
➤ Rhombus like Plane	
❖ Solid Angle subtended by 3-D Figures	10
➤ Sphere	
➤ Circular Cylinder	
➤ Right Circular Cone	
➤ Right Pyramid	
➤ Right Prism	
➤ Torus, Ellipsoid & Paraboloid	
❖ Solid State (arrangements of identical spheres)	16
➤ Simple Cubic Cell, B.C.C. Unit Cell, F.C.C. Unit Cell	
❖ Photometry/Radiometry	17
➤ Luminous Flux, Intensity, Radiation Energy, Lambert's Formula	
❖ Space Science (Celestial Bodies)	18
➤ Energy of Interception by Earth & Moon emitted by Sun	
❖ Analysis of Frustums	20
➤ Sphere, Right Circular Cone, Circular Cylinder	
❖ Analysis of Miscellaneous Articles	22
➤ Vector's Cosine Formula, HCR's Cosine Formula	
➤ Axioms of Tetrahedron, Concurrent Vectors, Triangle	
➤ Spiked Regular Polygon, Right Pyramid & Analogy of a Right Cone	
❖ HCR's Theory of Polygon (Graphical Method)	25
➤ Solid Angle subtended by a Polygonal Plane at any Point in the Space	
➤ Axiom of Polygon & Right Triangle	
➤ Comparison of Analytical & Graphical Methods	

❖ Fundamental & Integral Theorems of Solid Angle

1. **Solid angle:** “The **three dimensional angle** subtended by an **object** at a certain point in the space”. It is also known as **Angle of View** for 2-D & 3-D objects.

It is usually denoted by ' ω ' (**Omega**). Its unit is **Steradian** (in brief 'sr').

Mathematically, it is given by the following expression (**Basic Definition**)

$$\omega = \frac{\text{area of visible surface of the object}}{(\text{distance } (r) \text{ of any point on visible surface from the given point})^2}$$

Where, distance ' r ' from the given point (observer) to all the points on visible surface of the object is constant. Solid angle subtended by a point & a straight line is zero.

Although the distance ' r ' from the given point to all the points of visible surface is no more constant except in case of a spherical surface. But still the distance ' r ' may be assumed to be constant for an infinitesimal small area ' dA ' on the visible surface of the given object.

This concept derives the fundamental theorem which is the integration of solid angle subtended by infinitesimal small area ' dA ' on visible surface at the given point in the space.

2. The solid angle subtended by an **object** at a given **point (observer)** in the space is given by **Fundamental Theorem**

$$\omega = \int_s \frac{dA}{r^2} \quad \forall (0 \leq \omega \leq 4\pi)$$

Where, distance ' r ' is constant within limit i.e. distance (r) from the given point is constant for all the points of infinitesimal small area ' dA ' of visible surface of the object.

3. The solid angle subtended by a **right circular cone** with apex angle ' 2α ' at the apex point is given as

$$\omega = 2\pi(1 - \cos\alpha)$$

If normal height of the right cone is ' H ' & radius of the base is ' R ' then the apex angle ' 2α ' is related as

$$\cos\alpha = \frac{H}{\sqrt{H^2 + R^2}}$$

$$\therefore \omega = 2\pi \left(1 - \frac{H}{\sqrt{H^2 + R^2}} \right)$$

4. The solid angle subtended by an object at a given point is given by '**Integral Theorem**' (in form of **spherical co-ordinates**)

$$\omega = \int_{\varphi_1}^{\varphi_2} \left| \int_{\theta_1}^{\theta_2} \sin\theta \, d\theta \right| d\varphi \quad \forall \theta \in [0, \pi] \text{ \& } \varphi \in [-\pi, \pi]$$

Where, ' θ ' & ' φ ' are the parameters (variables) depending on the

- a. **Geometrical shape** of the object &
- b. **Orientation of shape** w.r.t. to the given point (observer) in the space

❖ **Solid Angle Subtended by Plane Figures**

- 5. The solid angle subtended by a **plane-figure**, in the **first quadrant** w.r.t. foot of perpendicular drawn to the plane from a given point in the space, is given by the '**HCR's Corollary of Plane-figure**' (proposed by HCR)

$$\omega = \int_0^\varphi \frac{f(h \tan\varphi) d\varphi}{\sqrt{(f(h \tan\varphi))^2 + h^2 \sec^2\varphi}} \quad \forall 0 \leq \varphi \leq \frac{\pi}{2}$$

Where, 'h' is the normal height of the point from the plane

$y = f(x) = f(h \tan\varphi) \neq (x = \text{constant})$ is the equation of boundary-curve of the given plane in the first quadrant w.r.t. foot of perpendicular assuming as the origin &

' φ ' is variable & $x = h \tan\varphi$.

Note: Foot of perpendicular drawn from the point must lie within the boundary of plane & $y = f(x)$ must be differential at each point of the curve in the first quadrant i.e. $y = f(x)$ is the equation of single-boundary of the given plane..

Standard formulae:

Standard formulae to be remembered are following

- 6. The solid angle subtended by a **right triangular plane** with base 'b' & perpendicular 'p' at any point lying at a height 'h' on the vertical axis passing through the **acute angled vertex** common to the **perpendicular 'p' & hypotenuse** is given as

$$\omega = \sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + p^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + p^2}} \right) \left(\frac{h}{\sqrt{h^2 + p^2}} \right) \right\} \dots \dots \dots (1)$$

If the vertical axis passes through the vertex common to the **base 'b' & hypotenuse**, the solid angle is given as

$$\omega = \sin^{-1} \left\{ \frac{p}{\sqrt{p^2 + b^2}} \right\} - \sin^{-1} \left\{ \left(\frac{p}{\sqrt{p^2 + b^2}} \right) \left(\frac{h}{\sqrt{h^2 + b^2}} \right) \right\} \dots \dots \dots (2)$$

The above formulae are **extremely useful** which can be used to find out the solid angle subtended by **any polygonal plane** at **any point in the space**. (HCR's Invention-2013)

7. The solid angle subtended by a **right triangular plane** with orthogonal sides 'a' & 'b' at any point lying at a height 'h' on the vertical axis passing through the **right angled vertex** is given as

$$\omega = \cos^{-1} \left\{ \frac{h(a^2\sqrt{h^2 + b^2} + b^2\sqrt{h^2 + a^2})}{h^2(a^2 + b^2) + a^2b^2} \right\}$$

8. The solid angle subtended by a **rectangular plane** with sides 'l' & 'b' at any point lying at a height 'h' on the vertical axis passing through the **centre** is given as

$$\omega = 4 \sin^{-1} \left(\frac{lb}{\sqrt{(l^2 + 4h^2)(b^2 + 4h^2)}} \right)$$

9. The solid angle subtended by a **rectangular plane** with sides 'l' & 'b' at any point lying at a height 'h' on vertical axis passing through **one of the vertices** is given as

$$\omega = \sin^{-1} \left(\frac{lb}{\sqrt{(l^2 + h^2)(b^2 + h^2)}} \right)$$

10. The solid angle subtended by a **square plane** with each side 'a' at any point lying at a height 'h' on the vertical axis passing through the **centre** is given as

$$\omega = 4 \sin^{-1} \left(\frac{a^2}{a^2 + 4h^2} \right)$$

11. The solid angle subtended by a **square plane** with each side 'a' at any point lying at a height 'h' on the vertical axis passing through **one of the vertices** is given as

$$\omega = \sin^{-1} \left(\frac{a^2}{a^2 + h^2} \right)$$

12. The solid angle subtended by a **regular polygon** having 'n' number of the sides each of length 'a', at any point lying at a height 'h' on the vertical axis passing through the centre is given as

$$\omega = 2\pi - 2n \sin^{-1} \left(\frac{2h \sin \frac{\pi}{n}}{\sqrt{4h^2 + a^2 \cot^2 \frac{\pi}{n}}} \right) \quad \forall n \in N \text{ \& } n \geq 3$$

13. The solid angle subtended by a **circular plane** with a radius 'R' at any point lying at a height 'h' on the vertical axis passing through the centre is given as

$$\omega = 2\pi \left(1 - \frac{h}{\sqrt{h^2 + R^2}} \right)$$

14. The solid angle subtended by an **infinitely long rectangular plane** having width 'b' at any point lying at a height 'h' on the vertical axis passing through the **centre** is given as

$$\omega = 4 \sin^{-1} \left(\frac{b}{\sqrt{b^2 + 4h^2}} \right)$$

15. The solid angle subtended by an **infinitely long rectangular plane** having width 'b' at any point lying at a height 'h' on the vertical axis passing through the **vertex** is given as

$$\omega = \sin^{-1} \left(\frac{b}{\sqrt{b^2 + h^2}} \right)$$

16. Solid angle subtended by a **triangular plane** at any point lying on the vertical axis passing through any vertex is determined by drawing a perpendicular from the same vertex to its opposite side i.e. by dividing the triangular plane into two right triangular planes & using standard formula (from eq(1) or (2) above) as follows

$$\omega = \sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + p^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + p^2}} \right) \left(\frac{h}{\sqrt{h^2 + p^2}} \right) \right\}$$

17. The solid angle subtended by a **regular pentagonal plane** with each side 'a' at any point lying at a normal height h from one of the vertices is given as

$$\omega = 2 \left[\frac{3\pi}{10} + \sin^{-1} \left(\frac{h \sin 18^\circ}{\sqrt{h^2 + a^2 \cos^2 18^\circ}} \right) - \sin^{-1} \left(\frac{h \cos 36^\circ}{\sqrt{h^2 + a^2 \cos^2 18^\circ}} \right) - \sin^{-1} \left(\frac{2h \sin 18^\circ}{\sqrt{4h^2 + a^2 \cot^2 18^\circ}} \right) \right]$$

$$\forall h \rightarrow 0 \Rightarrow \omega \rightarrow \frac{3\pi}{5} \text{ (Angle of vertex of regular pentagon)}$$

$$\forall a \rightarrow 0 \text{ or } h \rightarrow \infty \Rightarrow \omega \rightarrow 0$$

18. The solid angle subtended by a **regular hexagonal plane** with each side 'a' at any point lying at a normal height h from one of the vertices is given as

$$\omega = 2 \left[\frac{\pi}{3} + \sin^{-1} \left(\frac{h}{\sqrt{4h^2 + 3a^2}} \right) - \sin^{-1} \left(\frac{h\sqrt{3}}{\sqrt{4h^2 + 3a^2}} \right) - \sin^{-1} \left(\frac{h}{2\sqrt{h^2 + 3a^2}} \right) \right]$$

$$\forall h \rightarrow 0 \Rightarrow \omega \rightarrow \frac{2\pi}{3} \text{ (Angle of vertex of regular hexagon)}$$

$$\forall a \rightarrow 0 \text{ or } h \rightarrow \infty \Rightarrow \omega \rightarrow 0$$

19. The solid angle subtended by a **regular heptagonal plane** with each side 'a' at any point lying at a normal height h from one of the vertices is given as

$$\omega = 2 \left[\frac{5\pi}{14} + \sin^{-1} \left(\frac{h \cos \left(\frac{2\pi}{7} \right)}{\sqrt{h^2 + a^2 \sin^2 \left(\frac{2\pi}{7} \right)}} \right) \right. \\ \left. + \sin^{-1} \left(\frac{h \cos \left(\frac{3\pi}{7} \right)}{\sqrt{\left(h^2 + 4a^2 \cos^2 \left(\frac{\pi}{7} \right) \sin^2 \left(\frac{3\pi}{7} \right) \right)}} \right) \right. \\ \left. - \sin^{-1} \left(\frac{h \cos \left(\frac{\pi}{7} \right)}{\sqrt{h^2 + a^2 \sin^2 \left(\frac{2\pi}{7} \right)}} \right) - \sin^{-1} \left(\frac{2h \cos \left(\frac{3\pi}{7} \right)}{\sqrt{4h^2 + a^2 \tan^2 \left(\frac{3\pi}{7} \right)}} \right) \right. \\ \left. - \sin^{-1} \left(\frac{h \cos \left(\frac{2\pi}{7} \right)}{\sqrt{h^2 + 4a^2 \cos^2 \left(\frac{\pi}{7} \right) \sin^2 \left(\frac{3\pi}{7} \right)}} \right) \right]$$

$$\forall h \rightarrow 0 \Rightarrow \omega \rightarrow \frac{5\pi}{7} \text{ (Angle of vertex of regular heptagon)}$$

$$\forall a \rightarrow 0 \text{ or } h \rightarrow \infty \Rightarrow \omega \rightarrow 0$$

20. The solid angle subtended by a **regular octagonal plane** with each side 'a' at any point lying at a normal height h from one of the vertices is given as

$$\omega = 2 \left[\frac{3\pi}{8} + \sin^{-1} \left(\frac{h}{\sqrt{2h^2 + a^2}} \right) + \sin^{-1} \left(\frac{h}{\sqrt{(2 + \sqrt{2})(2h^2 + a^2(3 + 2\sqrt{2}))}} \right) \right. \\ \left. - \sin^{-1} \left(\frac{h\sqrt{2} \cos 22.5^\circ}{\sqrt{2h^2 + a^2}} \right) - \sin^{-1} \left(\frac{h}{\sqrt{2h^2 + a^2(3 + 2\sqrt{2})}} \right) \right. \\ \left. - \sin^{-1} \left(\frac{h}{\sqrt{(4 + 2\sqrt{2})(h^2 + a^2(3 + 2\sqrt{2}))}} \right) \right]$$

$$\forall h \rightarrow 0 \Rightarrow \omega \rightarrow \frac{3\pi}{4} \text{ (Angle of vertex of regular octagon)}$$

$$\forall a \rightarrow 0 \text{ or } h \rightarrow \infty \Rightarrow \omega \rightarrow 0$$

❖ Approximate Analysis of Solid Angle

21. **Approximate value** of the solid angle subtended by a **symmetrical plane** at any point in the space is given by '**Approximation-theorem**' as follows

$$\omega \cong 2\pi \left[1 - \frac{1}{\sqrt{\left(1 + \frac{A \cos \theta}{\pi r^2}\right)}} \right] \quad \forall \left(0 \leq \theta \leq \frac{\pi}{2}\right)$$

Where ' A ' → is the area of the given symmetrical plane

' θ ' → is the angle between the normal to the plane & the line joining the given point to the centre of the plane &

' r ' → is the distance of the given point from the centre of the plane

Limitations:

This formula is applicable only for the planes satisfying the following two conditions

I.) The plane should have at least two axes of symmetry i.e. a point of symmetry &

II.) The factor of circularity (F_c) should be close to the unity i.e. $F_c \geq 0.5$

If the above conditions are satisfied then the error involved in the results obtained from Approximation-formula will be minimum & permissible.

22. The solid angle subtended by an elliptical plane with major & minor axes ' a ' & ' b ' at any point lying at a normal distance ' r ' from the centre is given as

$$\omega \cong 2\pi \left[1 - \frac{1}{\sqrt{\left(1 + \frac{ab}{r^2}\right)}} \right]$$

Factor of circularity of elliptical plane,

$$F_c = \sqrt{1 - e^2} \quad \left[\forall (0 < F_c < 1) \quad \& \quad e = \sqrt{1 - \frac{b^2}{a^2}} \right]$$

23. The solid angle subtended by a circular plane with a radius ' R ' at any point lying at a normal distance ' r ' from the centre is given as

$$\omega = 2\pi \left[1 - \frac{1}{\sqrt{\left(1 + \frac{R^2}{r^2}\right)}} \right]$$

Factor of circularity of circular plane,

$$F_c = 1$$

24. The solid angle subtended by a regular polygonal plane with 'n' number of the sides each of length 'a' at any point lying at a normal distance 'r' from the centre is given as

$$\omega \cong 2\pi \left[1 - \frac{1}{\sqrt{1 + \left(\frac{na^2 \cot \frac{\pi}{n}}{4\pi r^2}\right)}} \right] \quad \forall (n \in N \ \& \ n \geq 3)$$

Factor of circularity of regular polygonal plane,

$$F_c = \cos \frac{\pi}{n} \quad \left[\forall n \in N \ \& \ n \geq 3 \Rightarrow \left(\frac{1}{2} \leq F_c < 1\right) \right]$$

Higher is the number of sides higher will be the factor of circularity of a regular polygon.

25. The solid angle subtended by a rectangular plane with sides 'l' & 'b' at any point lying at a normal distance 'r' from the centre is given as

$$\omega \cong 2\pi \left[1 - \frac{1}{\sqrt{\left(1 + \frac{lb}{\pi r^2}\right)}} \right]$$

Factor of circularity of rectangular plane,

$$F_c = \frac{b}{\sqrt{b^2 + l^2}} \quad \left[\forall b < l \Rightarrow (0 < F_c < 0.707) \right]$$

26. The solid angle subtended by a rhombus-like plane with diagonals 'd₁' & 'd₂' at any point lying at a normal distance 'r' from the centre is given as

$$\omega \cong 2\pi \left[1 - \frac{1}{\sqrt{\left(1 + \frac{d_1 d_2}{2\pi r^2}\right)}} \right]$$

Factor of circularity of rhombus-like plane,

$$F_c = \frac{d_1}{\sqrt{d_1^2 + d_2^2}} \quad \left[\forall d_1 < d_2 \Rightarrow (0 < F_c < 0.707) \right]$$

27. The solid angle subtended by a circular plane at any point in the space is given as

$$\omega \cong 2\pi \left[1 - \left\{ 1 + \left(\frac{R\sqrt{d^2 + R^2 - 2dR\sin\theta - h^2}}{h^2} \right)^2 \right\}^{-\frac{1}{2}} \right]$$

The value of h^2 is found out from the **auxiliary equation** given as follows

$$\Rightarrow (A^2 + B^2)h^4 + (2AC - B^2k^2)h^2 + C^2 = 0$$

Where

$$A = -2 \left(\frac{R\cos\theta}{(d^2 + R^2 - 2dR\sin\theta)^{\frac{3}{2}}} \right), \quad B = 2 \left(\frac{d - R\sin\theta}{(d^2 + R^2 - 2dR\sin\theta)^{\frac{3}{2}}} \right)$$

$$C = R\cos\theta \left(\frac{1}{\sqrt{d^2 + R^2 - 2dR\sin\theta}} - \frac{1}{\sqrt{d^2 + R^2 + 2dR\sin\theta}} \right) \quad \& \quad k = \sqrt{d^2 + R^2 - 2dR\sin\theta}$$

28. The solid angle subtended by an elliptical plane with major axis 'a', minor axis 'b' & eccentricity 'e', at a distance 'd' from the centre to a particular point from which it appears as a perfect circular plane is given as

$$\Rightarrow \left[\omega = 2\pi \left(1 - \frac{ad\cos\theta}{b\sqrt{a^2 + d^2 + 2adsin\theta}} \right) = 2\pi \left(1 - \frac{d\cos\theta}{\sqrt{1 - e^2}\sqrt{a^2 + d^2 + 2adsin\theta}} \right) \right]$$

$$\forall \left(0 \leq \theta < \frac{\pi}{2} \quad \& \quad d \geq 0 \right)$$

Where, ' θ ' is the angle between normal to the plane & the line joining the particular point to the centre of the plane.

❖ Solid Angle subtended by 3-D figures

* **Solid angle subtended by the Universe at any point in the space is 4π sr.**

Sphere:

29. The solid angle subtended by a **closed surface** at any point completely inside its boundary is always **4π sr** ($\forall 0 \leq \omega \leq 4\pi$).

30. The solid angle subtended at any point lying on the plane by a **cap** fully covering the plane is always **2π sr**

31. The solid angle subtended by a sphere with a radius 'R' at any point lying at a distance d from the centre is given as

$$\omega = 2\pi \left[1 - \frac{\sqrt{d^2 - R^2}}{d} \right] \quad \forall d \geq R$$

32. The solid angle subtended by a hemispherical shell with a radius 'R' at any internal point lying on the axis at a distance 'x' from the centre is given as

$$\omega = 2\pi \left[1 + \frac{x}{\sqrt{x^2 + R^2}} \right] \quad \forall 0 \leq x < R$$

33. The solid angle subtended by a hemispherical shell with a radius 'R' at any external point lying on the axis at a distance 'x' from the centre of circular-opening is given as

$$\omega = 2\pi \left[1 - \frac{x}{\sqrt{x^2 + R^2}} \right]$$

34. The solid angle subtended by a **hemispherical shell** at any point lying **inside the shell** on the plane of circular opening, is always **$2\pi sr$**
35. The solid angle subtended by a **hemispherical shell** at the **peak-point** of the surface lying inside the shell, is always **$3.414\pi \approx 10.726068 sr$** irrespective of the radius.
36. The solid angle subtended by a **hemispherical shell** at the **centre of mass** is always **$2.894\pi sr \approx 9.0931112 sr$** irrespective of the radius.

• **Circular Cylinder:**

37. The solid angle subtended by the **solid cylinder** at any point lying on the longitudinal axis at a distance 'x' from the centre of front-end is given as

$$\omega = 2\pi \left[1 - \frac{x}{\sqrt{x^2 + R^2}} \right]$$

38. The solid angle subtended by a cylinder with radius 'R' & length 'L' at any point lying on the transverse axis at a distance 'x' from the centre (neglecting the effect of curvature of circular ends) is given as

$$\omega = 4 \sin^{-1} \left[\frac{RL}{\sqrt{x^2 L^2 + 4(x^2 - R^2)^2}} \right] \quad \forall [x \geq R]$$

39. The solid angle subtended by an **infinitely long cylinder** with a radius 'R' at any point lying on the transverse axis at a distance 'x' from the centre is given as

$$\omega = 4 \sin^{-1} \left(\frac{R}{x} \right) \quad \forall [x \geq R]$$

40. The solid angle subtended by a hollow cylindrical shell with radius 'R' & length 'L' at any internal point lying on the longitudinal axis at a distance 'x' from the centre of one of the ends is given as

$$\omega = 2\pi \left[\frac{x}{\sqrt{x^2 + R^2}} + \frac{(L-x)}{\sqrt{(L-x)^2 + R^2}} \right] \quad \forall x \in [0, L]$$

41. The solid angle subtended by a hollow cylindrical shell with radius 'R' & length 'L' at any external point lying on the longitudinal axis at a distance 'x' from the centre of front end is given as

$$\omega = 2\pi \left[\frac{(L+x)}{\sqrt{(L+x)^2 + R^2}} - \frac{x}{\sqrt{x^2 + R^2}} \right] \quad \forall [x \geq 0]$$

42. The solid angle subtended by a **hollow cylindrical shell** with radius 'R' & length 'L' at the centre is given as

$$\omega = 4\pi \left[\frac{L}{\sqrt{L^2 + 4R^2}} \right]$$

• **Right Circular Cone:**

43. The solid angle subtended by a right cone with normal height 'H' & radius 'R' at the apex point is given as

$$\omega = 2\pi \left[1 - \frac{H}{\sqrt{H^2 + R^2}} \right]$$

44. The solid angle subtended by a hollow right conical shell with normal height 'H' & radius 'R' at any internal point lying on the axis at a distance 'x' from the centre of base is given as

$$\omega = 2\pi \left[1 + \frac{x}{\sqrt{x^2 + R^2}} \right] \quad \forall 0 \leq x < H$$

45. The solid angle subtended by a hollow right conical shell with normal height 'H' & radius 'R' at any external point lying on the axis at a distance 'x' from the centre of base is given as

$$\omega = 2\pi \left[1 - \frac{x}{\sqrt{x^2 + R^2}} \right]$$

46. The solid angle subtended by a hollow right conical shell with normal height 'H' & radius 'R' at the apex point inside the shell is given as

$$\omega = 2\pi \left[1 + \frac{H}{\sqrt{H^2 + R^2}} \right]$$

47. The solid angle subtended by a hollow right conical shell with normal height 'H' & radius 'R' at the centre of mass is given as

$$\omega = 2\pi \left[1 + \frac{H}{\sqrt{H^2 + 9R^2}} \right]$$

- **Right Pyramid:**

48. The solid angle subtended by a **right pyramid** with base as a **regular polygon** with 'n' number of the sides each of length 'a', normal height 'H' & angle between consecutive lateral edges ' α ', at any internal point lying on the axis at a distance 'x' from the centre of base is given as

$$\omega = 2\pi - 2n \sin^{-1} \left(\frac{2H \sin \frac{\pi}{n}}{\sqrt{4H^2 + a^2 \cot^2 \frac{\pi}{n}}} \right) = 2\pi - 2n \sin^{-1} \left(\cos \frac{\pi}{n} \sqrt{\tan^2 \frac{\pi}{n} - \tan^2 \frac{\alpha}{2}} \right)$$

Where, the relation between a , H , α & n is given as

$$H = \frac{a}{2} \sqrt{\cot^2 \frac{\alpha}{2} - \cot^2 \frac{\pi}{n}} \quad \forall \left(n \in N, n \geq 3 \text{ \& } 0 < \alpha \leq \frac{2\pi}{n} \right)$$

49. The solid angle subtended by a **right pyramidal shell** with base as a regular polygon with 'n' number of the sides each of length 'a' & normal height 'H', at any internal point lying on the axis at a distance 'x' from the centre of base is given as

$$\omega = 2\pi + 2n \sin^{-1} \left(\frac{2x \sin \frac{\pi}{n}}{\sqrt{4x^2 + a^2 \cot^2 \frac{\pi}{n}}} \right) \quad \forall 0 \leq x < H$$

50. The solid angle subtended by a **right pyramidal shell** with base as a regular polygon with 'n' number of the sides each of length 'a' & normal height 'H', at any external point lying on the axis at a distance 'x' from the centre of base is given as

$$\omega = 2\pi - 2n \sin^{-1} \left(\frac{2x \sin \frac{\pi}{n}}{\sqrt{4x^2 + a^2 \cot^2 \frac{\pi}{n}}} \right)$$

51. The solid angle subtended by a right pyramid with base as a regular polygon with 'n' number of the sides each of length 'a', normal height 'H' & angle between consecutive lateral edges ' α ' at apex point inside the shell is given as

$$\omega = 2\pi + 2n \sin^{-1} \left(\frac{2H \sin \frac{\pi}{n}}{\sqrt{4H^2 + a^2 \cot^2 \frac{\pi}{n}}} \right) = 2\pi + 2n \sin^{-1} \left(\cos \frac{\pi}{n} \sqrt{\tan^2 \frac{\pi}{n} - \tan^2 \frac{\alpha}{2}} \right)$$

52. The solid angle subtended by a **regular tetrahedron** at each of the vertices (**externally**), is **0.55 sr**.

The solid angle subtended by a **regular tetrahedron** at each of the vertices (**internally**), is **6.83 sr**.

53. The solid angle subtended by **each of the octants of orthogonal planes** at the origin point is 0.5π sr.

• **Right Prism:**

54. The solid angle subtended by a **solid right prism** with cross-section as a regular polygon with 'n' number of the sides each of length 'a' & length 'L', at any point lying on the longitudinal axis at a distance 'x' from the centre of front end is given as

$$\omega = 2\pi - 2n \sin^{-1} \left(\frac{2x \sin \frac{\pi}{n}}{\sqrt{4x^2 + a^2 \cot^2 \frac{\pi}{n}}} \right)$$

In this case, value of solid angle is **independent of the length of right prism.**

55. The solid angle subtended by a **hollow right prismatic shell** with cross-section as a regular polygon with 'n' number of the sides each of length 'a' & length 'L', at any **internal point** lying on the longitudinal axis at a distance 'x' from the centre of one of the ends is given as

$$\omega = 2n \left[\sin^{-1} \left(\frac{2x \sin \frac{\pi}{n}}{\sqrt{4x^2 + a^2 \cot^2 \frac{\pi}{n}}} \right) + \sin^{-1} \left(\frac{2(L-x) \sin \frac{\pi}{n}}{\sqrt{4(L-x)^2 + a^2 \cot^2 \frac{\pi}{n}}} \right) \right]$$

$$\forall x \in [0, L], \quad n \in N \text{ \& } n \geq 3$$

56. The solid angle subtended by a **hollow right prismatic shell** with cross-section as a regular polygon with 'n' number of the sides each of length 'a' & length 'L', at any **external point** lying on the longitudinal axis at a distance 'x' from the centre of front end is given as

$$\omega = 2n \left[\sin^{-1} \left(\frac{2(L+x) \sin \frac{\pi}{n}}{\sqrt{4(L+x)^2 + a^2 \cot^2 \frac{\pi}{n}}} \right) - \sin^{-1} \left(\frac{2x \sin \frac{\pi}{n}}{\sqrt{4x^2 + a^2 \cot^2 \frac{\pi}{n}}} \right) \right] \quad \forall x \geq 0$$

57. The solid angle subtended by a **hollow right prismatic shell** with cross-section as a regular polygon with 'n' number of the sides each of length 'a' & length 'L' at the **centre** is given as

$$\omega = 4n \sin^{-1} \left(\frac{L \sin \frac{\pi}{n}}{\sqrt{L^2 + a^2 \cot^2 \frac{\pi}{n}}} \right)$$

58. The solid angle subtended by a **hollow right prismatic shell** with cross-section as a regular polygon with 'n' number of the sides each of length 'a' & length 'L' at the **centre of one of the open ends** is given as

$$\omega = 2n \sin^{-1} \left(\frac{2L \sin \frac{\pi}{n}}{\sqrt{4L^2 + a^2 \cot^2 \frac{\pi}{n}}} \right)$$

59. The solid angle subtended by a **right pyramidal shell** with one end open & another closed, cross-section as a regular polygon with 'n' number of the sides each of length 'a' & length 'L', at any internal point lying on the longitudinal axis at a distance 'x' from the **centre of open end** is given as

$$\omega = 2\pi + 2n \sin^{-1} \left(\frac{2x \sin \frac{\pi}{n}}{\sqrt{4x^2 + a^2 \cot^2 \frac{\pi}{n}}} \right) \quad [\forall 0 \leq x < L, n \in \mathbb{N} \text{ \& } n \geq 3]$$

60. The solid angle subtended by a right pyramidal shell with one end open & another closed, cross-section as a regular polygon with 'n' number of the sides each of length 'a' & length 'L' at any internal point lying on the longitudinal axis at a distance 'x' from the centre of front end (open or closed) is given as

$$\omega = 2\pi - 2n \sin^{-1} \left(\frac{2x \sin \frac{\pi}{n}}{\sqrt{4x^2 + a^2 \cot^2 \frac{\pi}{n}}} \right) \quad [\forall x \geq 0]$$

• **Torus, Ellipsoid & Paraboloid:**

61. The solid angle subtended by a **torus** with inner & outer radii 'r' & 'R' respectively at any point lying at a height 'h' on the vertical axis passing through centre is given as

$$\omega = 4\pi \left(\frac{R^2 - r^2}{(R + r)^2 + 4h^2} \right) \quad \forall R \geq r \text{ \& } R, r, h \in [0, \infty)$$

62. The solid angle subtended by a **torus** with inner & outer radii 'r' & 'R' respectively at the centre is given as

$$\omega = 4\pi \left(\frac{R - r}{R + r} \right)$$

63. The solid angle (ω) subtended by an **ellipsoid** generated by rotating an ellipse $x^2/a^2 + y^2/b^2 = 1$ about the major axis at any point lying on the **major axis** at a distance d from the centre is given as

$$\omega = 2\pi \left(1 - \sqrt{\frac{(d^2 - a^2)}{(d^2 - a^2) + b^2}} \right) \quad \forall d \geq a$$

64. The solid angle (ω) subtended by an **ellipsoid** generated by rotating an ellipse $x^2/a^2 + y^2/b^2 = 1$ about the minor axis at any point lying on the **minor axis** at a distance d from the centre is given as

$$\omega = 2\pi \left(1 - \sqrt{\frac{(d^2 - b^2)}{(d^2 - b^2) + a^2}} \right) \quad \forall d \geq b$$

65. The solid angle (ω) subtended by a **paraboloid** generated by rotating the parabola $y^2 = 4ax$ about the axis at any point lying on the axis at a distance d from the **vertex** is given as

$$\omega = 2\pi \left(1 - \sqrt{\frac{d}{d+a}} \right) \quad \forall d \geq 0$$

❖ **Solid State (arrangement of identical spheres):**

66. The **solid angle** subtended by 'n' number of the **identical spheres** each with a radius 'R' arranged, touching one another, in a complete circular fashion at a **point lying on the vertical axis** at a height 'h' from the centre of reference plane is given as

$$\omega = 2n\pi \left[1 - \sqrt{\frac{h^2 + R^2 \cot^2 \frac{\pi}{n}}{h^2 + R^2 \operatorname{cosec}^2 \frac{\pi}{n}}} \right] \quad [\forall n \in N \ \& \ n \geq 3]$$

67. The **solid angle** subtended by 'n' number of the identical spheres each with a radius 'R' arranged, touching one another, in a complete circular fashion at the **centre of reference plane** is given as

$$\omega = 4n\pi \sin^2 \left(\frac{\pi}{2n} \right) \quad [\forall n \in N \ \& \ n \geq 3]$$

68. The solid angle subtended by the **spheres** forming a **simple cubic cell** at the centre is **9.223888 sr**. Each of the spheres subtends **1.152986 sr**.

69. The **ratio** of radius (r) of the largest sphere fully trapped in a **simple cubic cell** to the radius (R) of spheres forming the cell, is $r/R = 0.732051$

70. The **solid angle** subtended by the **largest sphere fully trapped** in a **simple cubic cell** at the centre of each of the spheres forming the cell is **0.588778 sr**.

71. The solid angle subtended by the spheres forming a **tetrahedral void** at the centre is **10.622346 sr**. Each of the spheres subtends **2.655586 sr**.

72. The **ratio** of radius (r) of the **largest sphere fully trapped** in a **tetrahedral void** to the radius (R) of spheres forming the void, is $r/R = 0.224745$
73. The **solid angle** subtended by the **largest sphere fully trapped** in a **tetrahedral void** at the centre of each of the spheres forming the void is **0.106694 sr**.
74. The solid angle subtended by the spheres forming a octahedral void at the centre is **11.041814 sr**. Each of the spheres subtends **1.840302 sr**.
75. The **ratio** of radius (r) of the **largest sphere fully trapped** in a **octahedral void** to the radius (R) of spheres forming the void, is $r/R = 0.414213$
76. The **solid angle** subtended by the **largest sphere fully trapped** in an **octahedral void** at the centre of each of the spheres forming the void is **0.275548 sr**.

* About **15.47%** of **tetrahedral void** & **12.13%** of **octahedral void** is **exposed** to their **outside (surrounding) space**. Thus an octahedral void is **more packed** than a tetrahedral void. (Mr HC Rajpoot)

❖ Photometry/Radiometry

77. The radiation (visible light or other E.M. radiation) energy emitted by uniform point-source located at the centre of a spherical surface is uniformly distributed over the entire the surface.
78. The **radiation** (visible light or other E.M. radiation) energy emitted by uniform linear-source coincident with the axis of a cylindrical surface is **uniformly** distributed over the entire the surface.
79. The **distribution of radiation energy**, emitted by a uniform point-source, over a plane surface is **no more constant**. Energy density at the foot of perpendicular drawn from the point-source to the plane is maximum & decreases successively at the points lying away from point of maxima.
80. The **total luminous flux**, emitted by a uniform point-source with luminous intensity I , intercepted by an object with visible surface 'S', is given as

$$F = I \times \omega = I \times \int_s \frac{dA}{r^2}$$

Where, ' ω ' is the solid angle subtended by the same object at the given point-source.

81. The **total radiation energy**, emitted by a uniform point-source with luminosity L , intercepted by an object with visible surface 'S', is given as

$$E = \left(\frac{L}{4\pi}\right) \times \omega = \left(\frac{L}{4\pi}\right) \int_s \frac{dA}{r^2}$$

Where, ' ω ' is the **solid angle** subtended by the same **object** at the given **point-source**.

82. **Density of luminous flux (or luminosity)** emitted by a uniform point-source with intensity I , over a plane-surface (S) with a total area 'A' (assuming uniform distribution of intercepted flux over the entire plane), is given as

$$F.D. = \frac{\text{Total flux (F) intercepted by the plane}}{\text{Area (A)}} = \frac{I \times \omega}{A} = \left(\frac{I}{A}\right) \int_s \frac{dA}{r^2}$$

83. **Lambert's cosine formula:**

Density of luminous flux (or luminosity (E)) emitted by a uniform point-source with intensity 'I' over a plane-surface with area 'A' (assuming uniform distribution of intercepted flux over the entire plane), is given

$$E = \frac{I \cos \theta}{r^2}$$

Where, 'r' is the distance of point-source from the centre of the plane &

' θ ' is the angle between normal to the plane & line joining point-source to the centre.

Limitation: The above formula is applicable for smaller plane surfaces and longer distances for minimum error involved in the results obtained.

❖ Space Science (Celestial Bodies):

84. The solid angle subtended by a **celestial body** with a mean radius at any point in the space (assuming that the body has a perfect spherical shape) is given as

$$\omega = 2\pi \left[1 - \frac{\sqrt{d^2 - R^2}}{d} \right] \quad (\forall d \geq R)$$

85. The solid angle subtended by the **Sun** at the centre of the **Earth** is about $6.79 \times 10^{-5} \text{ sr}$.

86. The solid angle subtended by the Earth at the centre of the Sun is about $5.71 \times 10^{-9} \text{ sr}$.

87. The solid angle subtended by the Sun at the centre of the Earth is about **11891.42** times that subtended by the Earth at the centre of the Sun.

88. The total radiation energy, emitted by the Sun, incident on the Earth is about $1.75 \times 10^{17} \text{ J/sec}$ & energy density over the intercepted surface of the Earth is

about 686.43 W/m^2 (Assuming no part of the radiation, emitted by the Sun, travelling to the Earth is absorbed or reflected).

89. The solid angle subtended by the Moon at the centre of the Earth is about $6.41 \times 10^{-5} \text{ sr}$.

90. The solid angle subtended by the Earth at the centre of the Moon is about 8.63×10^{-4} sr.
91. The solid angle subtended by the Earth at the centre of the Moon is about **13.45** times that subtended by the Moon at the centre of the Earth.
92. The luminosity (L) of a star with a mean radius ' R_o ', effective temperature ' T_e ' (absolute) & emissivity 'e' is given as

$$L = 4\pi e\sigma R_o^2 T_e^4$$

Where, $\sigma = 5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$

If the star is assumed as a perfect black body i.e. $e = 1$ then we have

$$L = 4\pi\sigma R_o^2 T_e^4$$

93. Brightness (B) of a star with a mean radius ' R_o ', effective temperature ' T_e ' (absolute) & emissivity 'e' at a point lying at a distance 'x' from the centre is given as

$$B = e\sigma T_e^4 \left(\frac{R_o}{x}\right)^2$$

If the star is assumed as a perfect black body i.e. $e = 1$ then we have

$$B = \sigma T_e^4 \left(\frac{R_o}{x}\right)^2$$

94. The relation between luminosity (L) & brightness (B) of a star at a particular point lying at a distance 'x' from the centre, is given as

$$B = \frac{L}{4\pi x^2}$$

95. Energy emitted by a uniform spherical-source with mean radius R_o , luminosity 'L', effective temperature ' T_e ' & emissivity 'e', intercepted by a spherical body with a radius 'R' lying at a distance 'd' between their centres, is given as

$$E_i = \left(\frac{L}{2}\right) \left(1 - \frac{\sqrt{d^2 - R^2}}{d}\right) = 2\pi e\sigma R_o^2 T_e^4 \left(1 - \frac{\sqrt{d^2 - R^2}}{d}\right)$$

If the spherical-source is assumed as a perfect black body i.e. emissivity, $e = 1$ then we have

$$E_i = \left(\frac{L}{2}\right) \left(1 - \frac{\sqrt{d^2 - R^2}}{d}\right) = 2\pi\sigma R_o^2 T_e^4 \left(1 - \frac{\sqrt{d^2 - R^2}}{d}\right)$$

96. **Area of interception** of a spherical body with a radius 'R' w.r.t. a given point lying at a distance d from the centre is given as

$$A_i = 2\pi R^2 \left(1 - \frac{R}{d}\right) \quad [\forall d \geq R]$$

97. If radiation energy, emitted by a uniform spherical-source with mean radius R_o , effective temperature ' T_e ', emissivity ' e ' & luminosity ' L ', is uniformly distributed over the area of interception of a spherical body with a radius ' R ' & lying at a distance d from the centre of the source then the **Energy Density (E.D.)** is given as

$$E.D. = \left(\frac{L}{4\pi R^2}\right) \left(\frac{d - \sqrt{d^2 - R^2}}{d - R}\right) = e\sigma T_e^4 \left(\frac{R_o}{R}\right)^2 \left(\frac{d - \sqrt{d^2 - R^2}}{d - R}\right) \quad [\forall d > (R_o + R)]$$

The above expression is applicable for uniform spherical-source like a star. If the source is assumed as perfect black body, take $e=1$ in the above expression.

❖ Analysis of Frustums

• Frustum of Sphere:

98. A frustum, of the sphere with a radius ' R ', has a single circular-section then the radius (r) of circular-section, surface area (A) & volume (V) are given as

$$\text{Radius, } r = \sqrt{2Rh - h^2}$$

$$\text{surface area, } A = 2\pi Rh \text{ \&}$$

$$\text{Volume, } V = \frac{\pi h^2}{3} (3R - h) \quad \forall (0 < h \leq 2R)$$

Where, ' h ' is the radial thickness of the frustum.

If the angle subtended by the circular-section of the frustum at the centre of parent sphere with radius ' R ' is 2θ then we have

$$\text{Radius, } r = R \sin \theta$$

$$\text{surface area, } A = 2\pi R^2 (1 - \cos \theta) \text{ \&}$$

$$\text{Volume, } V = \frac{\pi R^3}{12} (\cos 3\theta - 9\cos \theta + 8) \quad [\forall 0 < \theta \leq \pi]$$

99. A frustum, of the sphere with a radius ' R ', has two parallel circular-sections with radii r_1 & r_2 subtending angles θ_1 & θ_2 respectively at the centre of parent sphere then radii (r_1 & r_2) of circular-sections, surface area (A) & volume (V) of the frustum are given as

$$\text{Radii, } r_1 = R \sin \theta_1 \text{ \& } r_2 = R \sin \theta_2$$

$$\text{Area of the surface, } A = 2\pi R^2 (\cos \theta_1 - \cos \theta_2) \text{ \&}$$

$$\text{Volume, } V = \frac{\pi R^3}{12} [9(\cos\theta_1 - \cos\theta_2) - (\cos 3\theta_1 - \cos 3\theta_2)]$$

$$\forall (\theta_1 < \theta_2 \Rightarrow \theta_1 \in [0, \pi] \text{ \& } \theta_2 \in (0, \pi])$$

100. A frustum of the sphere with a radius 'R' has two parallel & identical circular-sections each subtending an angle 2θ at the centre of parent sphere then radius 'r' of circular-sections, surface area (A) & volume of the frustum are given as

$$\text{Radius, } r = R \sin\theta$$

$$\text{Area of the surface, } A = 4\pi R^2 \cos\theta \text{ \& }$$

$$\text{Volume, } V = \frac{\pi R^3}{12} (9\cos\theta - \cos 3\theta) \left[\forall 0 \leq \theta < \frac{\pi}{2} \right]$$

• **Frustum of Right Circular Cone:**

101. Major axis ($2a$), minor axis ($2b$) & eccentricity (e) of an elliptical section, generated by cutting a right cone with apex angle 2α with a smooth plane inclined at an angle θ with the axis & at a normal distance 'h' from the apex point, are given as

$$\text{major axis, } 2a = \frac{h \sin 2\alpha}{\sin^2 \theta - \sin^2 \alpha}$$

$$\text{minor axis, } 2b = \frac{2h \sin \alpha}{\sqrt{\sin^2 \theta - \sin^2 \alpha}} \text{ \& }$$

$$\text{eccentricity, } e = \frac{\cos \theta}{\cos \alpha} \quad \forall \left(0 < \theta \leq \frac{\pi}{2} \text{ \& } 0 < \alpha < \frac{\pi}{2} \Rightarrow \theta > \alpha \right)$$

102. **Angular shift (α_s):** Angle between axis of the right cone & line joining the centre of elliptical section (generated by cutting right cone obliquely with the axis) to the apex point. It is given as

$$\alpha_s = \tan^{-1}(\tan^2 \alpha \cot \theta) \quad \forall \left(0 < \theta \leq \frac{\pi}{2} \text{ \& } 0 < \alpha < \frac{\pi}{2} \Rightarrow \theta > \alpha \right)$$

103. **Perimeter (P) of elliptical section**

$$P = \left(\frac{2\pi h \tan \alpha}{\sin^2 \theta} \right) F_P$$

Where

$$\begin{aligned} F_P &= 1 + \sum_{n=1}^{\infty} \left[\left\{ \frac{1 \times 3 \times 5 \times \dots \times (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right] \\ &= 1 + 0.5C^2 + 0.375C^4 + 0.3125C^6 + 0.2734375C^8 + 0.24609375C^{10} \\ &\quad + 0.225585937C^{12} + 0.209472656C^{14} + \dots \dots \dots \infty \end{aligned}$$

104. **Volume (V) of oblique frustum with elliptical section**

$$V = \frac{\pi h^3 \sin^3 \theta \sin^2 \alpha \cos^4 \alpha}{(\sin^2 \theta - \sin^2 \alpha)^3} - \left(\frac{2\pi h^3 \tan^2 \alpha}{3 \sin^3 \theta} \right) F_V$$

Where

$$F_V = 1 + \sum_{n=1}^{\infty} \left[(n+1)(2n+1) \left\{ \frac{1 \times 3 \times 5 \times \dots \times (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right]$$

$$= 1 + 3C^2 + 5.625C^4 + 8.75C^6 + 12.3046875C^8 + 16.2421875C^{10}$$

$$+ 20.52832031C^{12} + 25.13671875C^{14} + \dots \dots \dots \infty$$

105. **Surface area (S) of oblique frustum with elliptical section**

$$S = \left(\frac{\pi h^2 \sec \alpha \tan \alpha}{\sin^2 \theta} \right) F_S$$

Where

$$F_S = 1 + \sum_{n=1}^{\infty} \left[(2n+1) \left\{ \frac{1 \times 3 \times 5 \times \dots \times (2n-3) \times (2n-1)}{2^n \times n!} \right\} C^{2n} \right]$$

$$= 1 + 1.5C^2 + 1.875C^4 + 2.1875C^6 + 2.4609375C^8 + 2.70703125C^{10}$$

$$+ 2.932617188C^{12} + 3.142089844C^{14} + \dots \dots \dots \infty$$

106. Radius of the circular section, generated by cutting a right cone with apex angle 2α with a smooth plane normal to the axis & at a normal distance 'h' from the apex point, is given as

$$R = 2h \tan \alpha \quad \forall \left(0 < \alpha < \frac{\pi}{2} \right)$$

• **Frustum of Circular Cylinder:**

107. Major axis (2a), minor axis (2b) & eccentricity (e) of an elliptical section, generated by cutting a circular cylinder with a diameter 'D' with a smooth plane inclined at an angle θ with the axis, are given as

$$\text{major axis, } 2a = D \operatorname{cosec} \theta$$

$$\text{minor axis, } 2b = D \quad \&$$

$$\text{eccentricity, } e = \cos \theta \quad \forall \left(0 < \theta \leq \frac{\pi}{2} \right)$$

❖ **Analysis of Miscellaneous Articles**

108. **Vectors' Cosine-formula:** (Derived by the author of the book)

If ' α ' is the angle between two concurrent vectors and one of them, keeping other stationary, is rotated by an angle ' β ' about the point of concurrency in a plane inclined at an angle ' θ '

with the plane containing both the vectors in initial position then the angle (α') between them in final position is given by the following expression

$$\Rightarrow \alpha' = \cos^{-1}(\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\theta) \quad \forall (0 \leq (\alpha, \theta) \leq \pi)$$

109. **HCR's Cosine-formula:**

If the angle α' between two concurrent vectors in the shifted position is known then the angle θ between the plane of rotation & the plane containing the vectors in initial position is given as

$$\theta = \cos^{-1}\left(\frac{\cos\alpha' - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right) \quad \forall (0 \leq (\alpha, \theta) \leq \pi)$$

110. **Axiom of tetrahedron:** If α, β & γ are the angles between consecutive lateral edges meeting at a vertex of a tetrahedron then the tetrahedron will exist only and only if the sum of any two angles out of α, β & γ is always greater than third one i.e.

$$(\alpha + \beta) > \gamma, (\beta + \gamma) > \alpha \text{ \& } (\alpha + \gamma) > \beta \quad \forall 0 < (\alpha + \beta + \gamma) < 2\pi$$

111. Internal angles θ_1, θ_2 & θ_3 , between lateral faces respectively opposite to the angles α, β & γ between consecutive lateral edges meeting at a vertex of a tetrahedron, are given as

$$\theta_1 = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right), \theta_2 = \cos^{-1}\left(\frac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}\right) \text{ \& } \\ \theta_3 = \cos^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right)$$

112. **Axiom of three concurrent-vectors:** The perpendicular, drawn from the point of concurrency of three equal vectors to the plane of a triangle generated by joining the heads of all the vectors, always passes through the circumscribed-centre of the triangle.

113. **Axiom of triangle:** The distance between circumscribed & inscribed centres in a triangle is given as

$$\Rightarrow D_{CI} = \sqrt{\left\{(r - R\cos\theta_{min})^2 + (R + r - R\cos\theta_{min})^2 \cot^2\left(\frac{2\theta + \theta_{min}}{2}\right)\right\}}$$

Where, $R \rightarrow$ radius of circumscribed circle

$r \rightarrow$ radius of inscribed circle

$\theta_{min} \rightarrow$ smallest angle &

$\theta \rightarrow$ any of the angles except θ_{min} in a given triangle

114. The distance, between the circumscribed & inscribed centres of an isosceles triangle having each of the equal angles θ , is given as

$$D_{CI} = \sqrt{(r - R\cos\theta)^2 + (R + r - R\cos\theta)^2 \cot^2\left(\frac{3\theta}{2}\right)} \quad \text{if } (\theta < (\pi - 2\theta))$$

$$D_{CI} = r + R\cos 2\theta \quad \text{if } (\theta > (\pi - 2\theta))$$

115. **Spiked/star-like regular-polygon:** Regular-polygon obtained by the straight lines joining each vertex to the end-points of its opposite side in a regular-polygon having odd no. of the sides is called a spiked regular-polygon or a star-like regular-polygon.

116. **Spike angle (2α)** of spiked regular polygon is given as

$$2\alpha = \frac{\pi}{n}$$

117. **Area (Δ)** of a spiked regular polygon is given as

$$\Delta = \frac{na^2}{4\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n}\right)} \quad \forall (n = 2m + 1 \Rightarrow m \in N)$$

118. **Radius** of the circle passing through tips of all the **spikes** of a spiked regular polygon with 'n' number of the spikes & span 'a', is given as

$$R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n} \quad \forall (n = 2m + 1 \Rightarrow m \in N)$$

119. **Right pyramid with base as a regular polygon:** Right pyramid with base as a regular polygon having 'n' number of the sides each of length 'a', angle between consecutive lateral edges ' α ' & normal height 'H'

Area of lateral surface,

$$A_l = \frac{1}{4} na \sqrt{4H^2 + a^2 \cot^2 \frac{\pi}{n}} = \frac{1}{4} na^2 \cot \frac{\alpha}{2}$$

Area of regular polygonal base,

$$A_b = \frac{1}{4} na^2 \cot \frac{\pi}{n}$$

Volume,

$$V = \frac{1}{12} na^2 H \cot \frac{\pi}{n} \quad \forall (n \in N \text{ \& } n \geq 3)$$

Relation between the angle α (between consecutive lateral edges) & acute angle β (between any of lateral edges & axis of the right pyramid)

$$\cot\beta = \sin\frac{\pi}{n} \sqrt{\cot^2\frac{\alpha}{2} - \cot^2\frac{\pi}{n}} \quad \forall \left(\alpha \leq \frac{2\pi}{n} \text{ \& } \beta \leq \frac{\pi}{2}\right)$$

Relation between normal height 'H' & length 'a' of side of the base

$$H = \frac{a}{2} \sqrt{\cot^2\frac{\alpha}{2} - \cot^2\frac{\pi}{n}}$$

120. **Analogy:**

Volume of a **right conical body** with **cross-section** varying **linearly** with its normal height (length) is given as

$$V = \frac{1}{3}(\text{area of base})(\text{normal height})$$

121. **Angle** (α) between any two bonds in a molecule having **regular tetrahedral** structure (like **CH₄ molecule**) is given as

$$\alpha = 2 \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 109.4712206^\circ \approx 109.47^\circ$$

122. Angle between any two lines joining the centre of a **tetrahedral void** to the centres of spheres (forming the void) is equal to $2 \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 109.47^\circ$

Note: In some of the books, it is given equal to 109.28° which is very closer to the most correct value **109.47^o**

❖ HCR's Theory of Polygon

• Axiom of Polygon

For a given point in the space, each of the polygons can be divided internally or externally or both (w.r.t. F.O.P.) into a certain number of the elementary triangles all having a common vertex at the foot of perpendicular (F.O.P.) drawn from the given point to the plane of polygon, by joining all the vertices of polygon to the F.O.P. by extended straight lines.

1. Polygon is externally divided if the F.O.P. lies outside the boundary
2. Polygon is divided internally or externally or both if the F.O.P. lies inside or on the boundary depending on the geometrical shape of polygon (i.e. angles & sides)

(See the figures (1), (2), (3), (4), (5) & (6) below)

• Axiom of Right Triangle

If the perpendicular, drawn from a given point in the space to the plane of a given triangle, passes through one of the vertices then the triangle can be divided internally or externally (w.r.t. F.O.P.) into two elementary right triangles having common vertex at the foot of perpendicular, by drawing a normal from the common vertex (i.e. F.O.P.) to the opposite side of given triangle.

1. An acute angled triangle is internally divided w.r.t. F.O.P. (i.e. common vertex)
2. A right angled triangle is internally divided w.r.t. F.O.P. (i.e. common vertex)
3. An obtuse angled triangle is divided
 - a. Internally if and only if the angle of common vertex (F.O.P.) is obtuse
 - b. Externally if and only if the angle of common vertex (F.O.P.) is acute

(As shown in the figures (1) & (2) below)

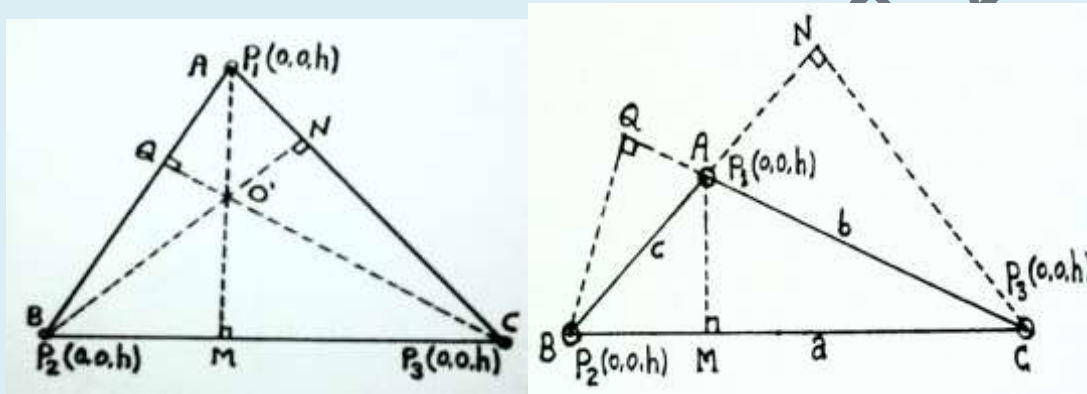


Fig 1: P_1, P_2 & P_3 are different locations Fig 2: P_1, P_2 & P_3 are different locations

- **Graphical Method: (For polygonal-planes)**

This method is based on above two axioms (proposed by the author) for analysis of polygonal planes. It is the most versatile method for finding out the solid angle subtended by a polygonal plane at any point in the space. This method overcomes all the limitations of Approximation formula used for symmetrical planes. Although, it necessitates the drawing of a given polygonal plane but it involves **theoretically zero error**. Still due to computational error some numerical error may be involved in the results obtained by this method.

Graphical method eliminates a lot of calculations in finding out the solid angle subtended by a given plane if the location of foot of perpendicular, drawn from a given point in the space, is known w.r.t. the given plane.

- **Working Steps:**

STEP 1: Trace/draw the diagram of a given polygon (plane) with known geometrical dimensions.

STEP 2: Specify the location of foot of perpendicular drawn from a given point to the plane of polygon.

STEP 3: Join all the vertices of polygon to the foot of perpendicular by the extended straight lines. Thus the polygon is divided into a number of elementary triangles, all having a common vertex at the foot of perpendicular.

STEP 4: Now, consider each elementary triangle & divide it into two sub-elementary right triangles to find the solid angle subtended by each elementary triangle.

STEP 5: Solid angle subtended by each individual sub-elementary right triangle is determined using the **standard formula of right triangle** given as

$$\omega = \sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + p^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + p^2}} \right) \left(\frac{h}{\sqrt{h^2 + p^2}} \right) \right\}$$

STEP 6: Now, solid angle, subtended by polygonal plane at the given point, will be the algebraic sum of solid angles of its individual sub-elementary right triangles as given

$$\omega_o = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \dots \dots \dots$$

Let's us consider a polygonal plane (with vertices) 123456 & a given point say $P(0, 0, h)$ at a normal height h from the given plane (i.e. plane of paper) in the space.

Now, specify the location of foot of perpendicular say 'O' on the plane of polygon which may lie ($P(0, 0, h)$ shows point P lies at a normal height h from the plane of paper)

1. Outside the boundary
2. Inside the boundary
3. On the boundary
 - a. On one of the sides or b. At one of the vertices

Let's consider the above cases one by one as follows

1. F.O.P. outside the boundary:

Let the foot of perpendicular 'O' lie outside the boundary of polygon. Join all the vertices of polygon (plane) 123456 to the foot of perpendicular 'O' by the extended straight lines

(As shown in the figure 3)

Thus the polygon is divided into elementary triangles (obtained by the extension lines), all having common vertex at the foot of perpendicular 'O'. Now the solid angle subtended by the polygonal plane at the given point 'P' in the space is given

By Element-Method

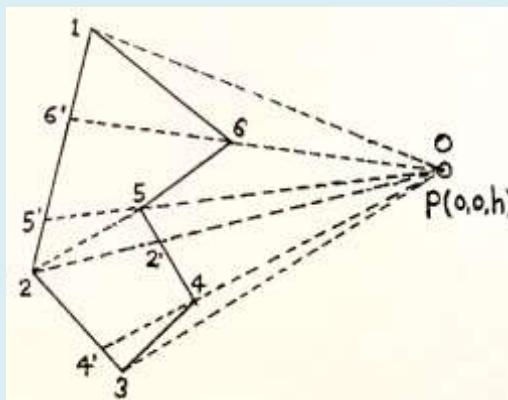


Figure 3: F.O.P. lying outside the boundary

$$\omega_{123456} = \omega_{616'} + \omega_{566'5'} + \omega_{22'55'} + \omega_{4'42'2} + \omega_{344'} \dots \dots \dots (I)$$

Where,

$$\omega_{616'} = \omega_{6'01} - \omega_{601}$$

$$\omega_{566'5'} = \omega_{5'06'} - \omega_{506} = \omega_{205'} - \omega_{2'05}$$

$$\omega_{4'42'2} = \omega_{4'02} - \omega_{402'}$$

$$\omega_{344'} = \omega_{304'} - \omega_{304}$$

Thus, the value of solid angle subtended by the polygon at the given point is obtained by setting these values in the eq. (I) as follows

$$\Rightarrow \omega_{123456} = (\omega_{6'01} - \omega_{601}) + (\omega_{5'06'} - \omega_{506}) + (\omega_{205'} - \omega_{2'05}) + (\omega_{4'02} - \omega_{402'})$$

Further, each of the individual elementary triangles is easily be divided into two sub-elementary right triangles for which the values of solid angle are determined by using standard formula of right triangle.

2. F.O.P. inside the boundary:

Let the foot of perpendicular 'O' lie inside the boundary of polygon. Join all the vertices of polygon (plane) 123456 to the foot of perpendicular 'O' by the extended straight lines

(As shown in the figure 4)

Thus the polygon is divided into elementary triangles (obtained by the extension lines), all having common vertex at the foot of perpendicular 'O'. Now the solid angle subtended by the polygonal plane at the given point P in the space is given

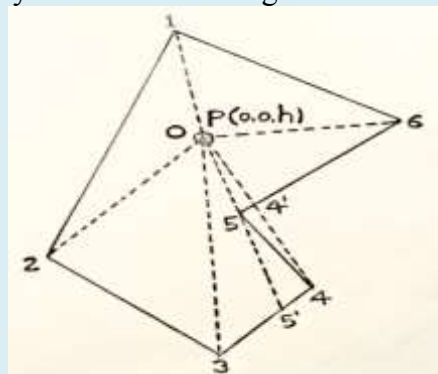


Figure 4: F.O.P. lying inside the boundary

By Element-Method

$$\omega_{123456} = \omega_{102} + \omega_{203} + \omega_{305'} + \omega_{5'54} + \omega_{504'} \dots \dots \dots (II)$$

Where,

$$\omega_{5'54} = \omega_{5'04} - \omega_{504}$$

Thus, the value of solid angle subtended by the polygon at the given point is obtained by setting these values in the eq. (II) as follows

$$\Rightarrow \omega_{123456} = \omega_{102} + \omega_{203} + \omega_{305'} + (\omega_{5'04} - \omega_{504}) + \omega_{504'}$$

Further, each of the individual elementary triangles is easily divided into two sub-elementary right triangles for which the values of solid angle are determined by using standard formula of right triangle.

3. F.O.P. on the boundary: Further two cases are possible

a. F.O.P. lying on one of the sides:

Let the foot of perpendicular ‘O’ lie on one of the sides say ‘12’ of polygon. Join all the vertices of polygon (plane) 123456 to the foot of perpendicular ‘O’ by the straight lines

(As shown in the figure 5)

Thus the polygon is divided into elementary triangles (obtained by the straight lines), all having common vertex at the foot of perpendicular ‘O’. Now the solid angle subtended by the polygonal plane at the given point ‘P’ in the space is given By Element-Method

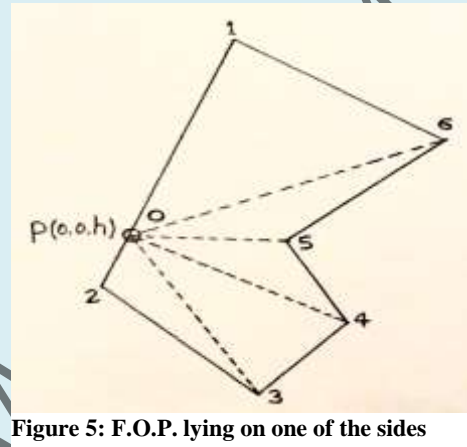


Figure 5: F.O.P. lying on one of the sides

$$\omega_{123456} = \omega_{106} + \omega_{605} + \omega_{504} + \omega_{403} + \omega_{302} \dots \dots \dots (III)$$

Further, each of the individual elementary triangles is easily divided into two sub-elementary right triangles for which the values of solid angle are determined by using formula of right triangle.

b. F.O.P. lying at one of the vertices:

Let the foot of perpendicular lie on one of the vertices say ‘5’ of polygon. Join all the vertices of polygon (plane) 123456 to the foot of perpendicular (i.e. common vertex ‘5’) by the straight lines

(As shown in the figure 6)

Thus the polygon is divided into elementary triangles

(obtained by the straight lines), all having common vertex at the foot of perpendicular ‘5’. Now the solid angle subtended by the polygonal plane at the given point ‘P’ is given

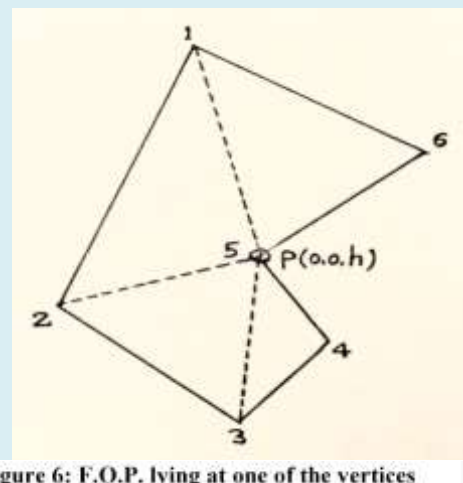


Figure 6: F.O.P. lying at one of the vertices

By Element-Method

$$\omega_{123456} = \omega_{152} + \omega_{253} + \omega_{354} + \omega_{456} \dots \dots \dots (IV)$$

Further, each of the individual elementary triangles is easily divided into two sub-elementary right triangles for which the values of solid angle are determined by using standard formula of right triangle.

• **Comparison of Approximation Method & Graphical Method**

Approximation Method	Graphical Method
1. It is applicable only for symmetrical plane-figures. 2. It is not useful for finding the precise/exact value of solid angle i.e. a certain error is included in the results. 3. It depends on the geometrical shape of symmetrical plane & its configuration (factor of circularity). 4. It requires mere mathematical calculations which are easier as compared to that of graphical method.	1. It is applicable only for polygonal plane-figures. 2. It gives theoretical zero error in the values if computations/calculations are done correctly. 3. It necessitates exact location of foot perpendicular (F.O.P.) drawn from given point to the plane which is not easy to specify. 4. It necessitates drawing since mathematical calculations are easy to find all the dimensions of elementary planes.

Illustrative Examples of HCR's Theory of Polygon

Example 1: Let's find out solid angle subtended by a pentagonal plane ABCDE having sides $AB = 6cm, BC = 5.7cm, CD = 6.65cm, DE = 5.5cm, AE = 7.8cm, \angle BAE = 120^\circ$ & $\angle ABC = 80^\circ$ at a point 'P' lying at a normal height 6cm from a point 'O' internally dividing the side AB such that $OA:OB = 2:1$ & calculate the total luminous flux intercepted by the plane ABCDE if a uniform point-source of 1400 lm is located at the point 'P'

Sol: Draw the pentagon ABCDE with known values of the sides & angles & specify the location of given point P by $P(0,0,6)$ perpendicularly outwards to the plane of paper & F.O.P. 'O' (as shown in the figure 7)

Divide the pentagon ABCDE into elementary triangles $\triangle OBC, \triangle OCD, \triangle ODE$ & $\triangle OEA$ by joining all the vertices of pentagon ABCDE to the F.O.P. 'O'. Further divide each of the triangles $\triangle OBC, \triangle OCD, \triangle ODE$ & $\triangle OEA$ in two right triangles simply by drawing a perpendicular to the opposite side in the respective triangle. (See the diagram)

It is clear from the diagram, the solid angle subtended by pentagonal plane ABCDE at the point 'P' is given by Element Method as follows

$$\omega_{ABCDE} = \omega_{\triangle OBC} + \omega_{\triangle OCD} + \omega_{\triangle ODE} + \omega_{\triangle OEA} \dots \dots (I)$$

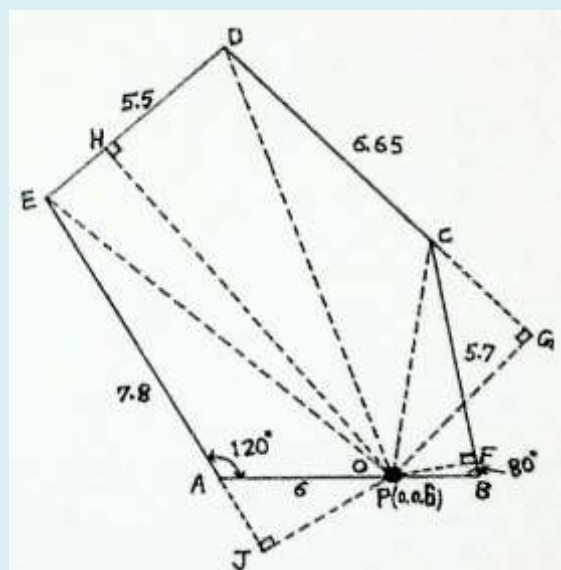


Fig 7: Point P is lying perpendicularly outwards to the plane of paper. All the dimensions are in cm.

From the diagram, it's obvious that the solid angle ω_{ABCDE} subtended by the pentagon ABCDE is expressed as the algebraic sum of solid angles of sub-elementary right triangles only as follows

$$\begin{aligned}\omega_{\Delta OBC} &= \omega_{\Delta OFB} + \omega_{\Delta OFC} & \omega_{\Delta OCD} &= \omega_{\Delta OGD} - \omega_{\Delta OGC} \\ \omega_{\Delta ODE} &= \omega_{\Delta OHD} + \omega_{\Delta OHE} & \omega_{\Delta OEA} &= \omega_{\Delta OJE} - \omega_{\Delta OJA}\end{aligned}$$

Now, setting the values in eq(I), we get

$$\omega_{ABCDE} = (\omega_{\Delta OFB} + \omega_{\Delta OFC}) + (\omega_{\Delta OGD} - \omega_{\Delta OGC}) + (\omega_{\Delta OHD} + \omega_{\Delta OHE}) + (\omega_{\Delta OJE} - \omega_{\Delta OJA}) \dots \dots \dots (II)$$

Now, measure the necessary dimensions & set them into standard formula-1 to find out above values of solid angle subtended by the sub-elementary right triangles at the given point 'P' as follows

$$\begin{aligned}\Rightarrow \omega_{\Delta OFB} &= \sin^{-1} \left\{ \frac{FB}{\sqrt{(FB)^2 + (OF)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{FB}{\sqrt{(FB)^2 + (OF)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OF)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{0.35}{\sqrt{(0.35)^2 + (1.95)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{0.35}{\sqrt{(0.35)^2 + (1.95)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (1.95)^2}} \right) \right\} \\ &= 0.177596167 - 0.168814218 = \mathbf{0.008781949 \text{ sr}}\end{aligned}$$

$$\begin{aligned}\Rightarrow \omega_{\Delta OFC} &= \sin^{-1} \left\{ \frac{FC}{\sqrt{(FC)^2 + (OF)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{FC}{\sqrt{(FC)^2 + (OF)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OF)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{5.35}{\sqrt{(5.35)^2 + (1.95)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{5.35}{\sqrt{(5.35)^2 + (1.95)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (1.95)^2}} \right) \right\} \\ &= 1.221275136 - 1.105149872 = \mathbf{0.116125264 \text{ sr}}\end{aligned}$$

$$\begin{aligned}\Rightarrow \omega_{\Delta OGD} &= \sin^{-1} \left\{ \frac{DG}{\sqrt{(DG)^2 + (OG)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{DG}{\sqrt{(DG)^2 + (OG)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OG)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{9.9}{\sqrt{(9.9)^2 + (4.6)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{9.9}{\sqrt{(9.9)^2 + (4.6)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (4.6)^2}} \right) \right\} \\ &= 1.135829376 - 0.803382922 = \mathbf{0.332446454 \text{ sr}}\end{aligned}$$

$$\begin{aligned}\Rightarrow \omega_{\Delta OGC} &= \sin^{-1} \left\{ \frac{CG}{\sqrt{(CG)^2 + (OG)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CG}{\sqrt{(CG)^2 + (OG)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OG)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{3.25}{\sqrt{(3.25)^2 + (4.6)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{3.25}{\sqrt{(3.25)^2 + (4.6)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (4.6)^2}} \right) \right\} \\ &= 0.615089573 - 0.475672072 = \mathbf{0.139417501 \text{ sr}}\end{aligned}$$

$$\begin{aligned}\Rightarrow \omega_{\Delta OHD} &= \sin^{-1} \left\{ \frac{DH}{\sqrt{(DH)^2 + (OH)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{DH}{\sqrt{(DH)^2 + (OH)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OH)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{3.7}{\sqrt{(3.7)^2 + (10.25)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{3.7}{\sqrt{(3.7)^2 + (10.25)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (10.25)^2}} \right) \right\} \\ &= 0.346418989 - 0.172376751 = \mathbf{0.174042238 \text{ sr}}\end{aligned}$$

$$\begin{aligned}
\Rightarrow \omega_{\Delta OHE} &= \sin^{-1} \left\{ \frac{EH}{\sqrt{(EH)^2 + (OH)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{EH}{\sqrt{(EH)^2 + (OH)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OH)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{1.8}{\sqrt{(1.8)^2 + (10.25)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{1.8}{\sqrt{(1.8)^2 + (10.25)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (10.25)^2}} \right) \right\} \\
&= 0.173837242 - 0.087488889 = \mathbf{0.086348353 \text{ sr}} \\
\Rightarrow \omega_{\Delta OJE} &= \sin^{-1} \left\{ \frac{EJ}{\sqrt{(EJ)^2 + (OJ)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{EJ}{\sqrt{(EJ)^2 + (OJ)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OJ)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{9.8}{\sqrt{(9.8)^2 + (3.45)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{9.8}{\sqrt{(9.8)^2 + (3.45)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (3.45)^2}} \right) \right\} \\
&= 1.232304566 - 0.957430258 = \mathbf{0.274874308 \text{ sr}} \\
\Rightarrow \omega_{\Delta OJA} &= \sin^{-1} \left\{ \frac{AJ}{\sqrt{(AJ)^2 + (OJ)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AJ}{\sqrt{(AJ)^2 + (OJ)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OJ)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{2}{\sqrt{(2)^2 + (3.45)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{2}{\sqrt{(2)^2 + (3.45)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (3.45)^2}} \right) \right\} \\
&= 0.525366873 - 0.449793847 = \mathbf{0.075573026 \text{ sr}}
\end{aligned}$$

Hence, by setting the corresponding values in eq(II), solid angle subtended by the pentagonal plane at the given point 'P' is calculated as follows

$$\begin{aligned}
\omega_{ABCDE} &= (\omega_{\Delta OFB} + \omega_{\Delta OFC}) + (\omega_{\Delta OGD} - \omega_{\Delta OGC}) + (\omega_{\Delta OHD} + \omega_{\Delta OHE}) + (\omega_{\Delta OJE} - \omega_{\Delta OJA}) \\
&= (0.008781949 + 0.116125264) + (0.332446454 - 0.139417501) + (0.174042238 + 0.086348353) \\
&\quad + (0.274874308 - 0.075573026) = \mathbf{0.777628039 \text{ sr}}
\end{aligned}$$

Calculation of Luminous Flux: If a **uniform point-source** of 1400 lm is located at the given point 'P' then the **total luminous flux intercepted** by the pentagonal plane ABCDE

$$= \frac{\text{solid angle} \times \text{total flux emitted by source}}{4\pi} = \frac{0.777628039 \times 1400}{4\pi} = \mathbf{86.63434241 \text{ lm (Lumen)}}$$

It means that only 86.63434241 lm out of 1400 lm flux is striking the pentagonal plane ABCDE & rest of the flux is escaping to the surrounding space. **This result can be experimentally verified. (H.C. Rajpoot)**

Example 2: Let's find out solid angle subtended by a quadrilateral plane ABCD having sides $AB = 8\text{cm}$, $BC = 9\text{cm}$, $CD = 6\text{cm}$, $AD = 4\text{cm}$ & $\angle BAD = 70^\circ$ at a point 'P' lying at a normal height 4cm from a point 'O' outside the quadrilateral ABCD such that $OB = 8\text{cm}$ & $OC = 4.2\text{cm}$ & calculate the total luminous flux intercepted by the plane ABCD if a uniform point-source of 1400 lm is located at the point 'P'

Sol: Draw the quadrilateral ABCD with known values of the sides & angle & specify the location of given point P by $P(0, 0, 4)$ perpendicularly outwards to the plane of paper & F.O.P. 'O' (See the figure 8)

Divide quadrilateral ABCD into elementary triangles $\Delta OAB, \Delta ODA$ & ΔOCD by joining all the vertices of quadrilateral ABCD to the F.O.P. 'O'. Further divide each of the triangles $\Delta OAB, \Delta ODA$ & ΔOCD in two right triangles simply by drawing perpendiculars OE, OG & OF to the opposite sides AB, AD & CD in the respective triangles. (See the diagram)

It is clear from the diagram, the solid angle subtended by quadrilateral plane ABCD at the given point 'P' is given by **Element Method** as follows

Area of quadrilateral ABCD = algebraic sum of areas of elementary triangles

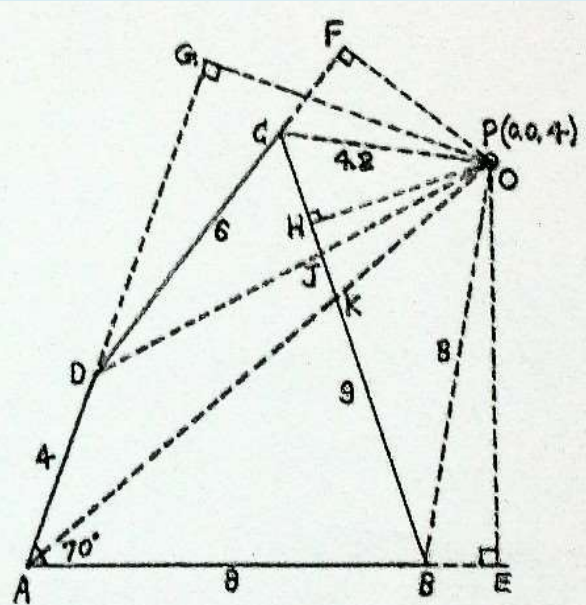


Fig 8: Point P is lying perpendicularly outwards to the plane of paper. All the dimensions are in cm.

$$\therefore A_{ABCD} = (A_{\Delta OAB} - A_{\Delta OKB}) + (A_{\Delta ODA} - A_{\Delta OJK}) + (A_{\Delta OCD} - A_{\Delta OCJ})$$

Now, replacing areas by corresponding values of solid angle, we get

$$\omega_{ABCD} = (\omega_{\Delta OAB} - \omega_{\Delta OKB}) + (\omega_{\Delta ODA} - \omega_{\Delta OJK}) + (\omega_{\Delta OCD} - \omega_{\Delta OCJ}) \dots \dots (I)$$

Now, draw a perpendicular OH from F.O.P. to the side BC to divide $\Delta OKB, \Delta OJK$ & ΔOCJ into right triangles & express the above values of solid angle as the algebraic sum of solid angles subtended by the right triangles only as follows

$$\begin{aligned} \omega_{\Delta OAB} &= \omega_{\Delta OEA} - \omega_{\Delta OEB} & \omega_{\Delta ODA} &= \omega_{\Delta OGA} - \omega_{\Delta OGD} & \omega_{\Delta OCD} &= \omega_{\Delta OFD} - \omega_{\Delta OFC} \\ \omega_{\Delta OKB} &= \omega_{\Delta OHB} - \omega_{\Delta OHK} & \omega_{\Delta OJK} &= \omega_{\Delta OHK} - \omega_{\Delta OHJ} & \omega_{\Delta OCJ} &= \omega_{\Delta OHC} + \omega_{\Delta OHJ} \end{aligned}$$

Now, setting the above values in eq(I), we get

$$\omega_{ABCD} = (\omega_{\Delta OEA} - \omega_{\Delta OEB} - \omega_{\Delta OHB} + \omega_{\Delta OHK}) + (\omega_{\Delta OGA} - \omega_{\Delta OGD} - \omega_{\Delta OHK} + \omega_{\Delta OHJ}) + (\omega_{\Delta OFD} - \omega_{\Delta OFC} - \omega_{\Delta OHC} - \omega_{\Delta OHJ})$$

$$\omega_{ABCD} = \omega_{\Delta OEA} + \omega_{\Delta OGA} + \omega_{\Delta OFD} - \omega_{\Delta OEB} - \omega_{\Delta OHB} - \omega_{\Delta OGD} - \omega_{\Delta OFC} - \omega_{\Delta OHC} \dots \dots (II)$$

Now, measure the necessary dimensions & set them into standard formula-1 to find out above values of solid angle subtended by the sub-elementary right triangles at the given point 'P' as follows

$$\begin{aligned} \Rightarrow \omega_{\Delta OEA} &= \sin^{-1} \left\{ \frac{AE}{\sqrt{(AE)^2 + (OE)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AE}{\sqrt{(AE)^2 + (OE)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OE)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{9.4}{\sqrt{(9.4)^2 + (7.9)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{9.4}{\sqrt{(9.4)^2 + (7.9)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (7.9)^2}} \right) \right\} \\ &= 0.871887063 - 0.353107934 = \mathbf{0.518779129 \text{ sr}} \end{aligned}$$

$$\Rightarrow \omega_{\Delta OGA} = \sin^{-1} \left\{ \frac{AG}{\sqrt{(AG)^2 + (OG)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AG}{\sqrt{(AG)^2 + (OG)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OG)^2}} \right) \right\}$$

$$= \sin^{-1} \left\{ \frac{10.5}{\sqrt{(10.5)^2 + (6)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{10.5}{\sqrt{(10.5)^2 + (6)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (6)^2}} \right) \right\}$$

$$= 1.051650213 - 0.502496173 = \mathbf{0.54915404 \text{ sr}}$$

$$\Rightarrow \omega_{\Delta OFD} = \sin^{-1} \left\{ \frac{DF}{\sqrt{(DF)^2 + (OF)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{DF}{\sqrt{(DF)^2 + (OF)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OF)^2}} \right) \right\}$$

$$= \sin^{-1} \left\{ \frac{8.1}{\sqrt{(8.1)^2 + (3.65)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{8.1}{\sqrt{(8.1)^2 + (3.65)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (3.65)^2}} \right) \right\}$$

$$= 1.147429185 - 0.738889475 = \mathbf{0.408539709 \text{ sr}}$$

$$\Rightarrow \omega_{\Delta OEB} = \sin^{-1} \left\{ \frac{BE}{\sqrt{(BE)^2 + (OE)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BE}{\sqrt{(BE)^2 + (OE)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OE)^2}} \right) \right\}$$

$$= \sin^{-1} \left\{ \frac{1.4}{\sqrt{(1.4)^2 + (7.9)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{1.4}{\sqrt{(1.4)^2 + (7.9)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (7.9)^2}} \right) \right\}$$

$$= 0.17539422 - 0.078906232 = \mathbf{0.096487988 \text{ sr}}$$

$$\Rightarrow \omega_{\Delta OHB} = \sin^{-1} \left\{ \frac{BH}{\sqrt{(BH)^2 + (OH)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BH}{\sqrt{(BH)^2 + (OH)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OH)^2}} \right) \right\}$$

$$= \sin^{-1} \left\{ \frac{7.1}{\sqrt{(7.1)^2 + (3.8)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{7.1}{\sqrt{(7.1)^2 + (3.8)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (3.8)^2}} \right) \right\}$$

$$= 1.079378081 - 0.69346571 = \mathbf{0.385912371 \text{ sr}}$$

$$\Rightarrow \omega_{\Delta OGD} = \sin^{-1} \left\{ \frac{DG}{\sqrt{(DG)^2 + (OG)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{DG}{\sqrt{(DG)^2 + (OG)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OG)^2}} \right) \right\}$$

$$= \sin^{-1} \left\{ \frac{6.5}{\sqrt{(6.5)^2 + (6)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{6.5}{\sqrt{(6.5)^2 + (6)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (6)^2}} \right) \right\}$$

$$= 0.82537685 - 0.419819479 = \mathbf{0.405557371 \text{ sr}}$$

$$\Rightarrow \omega_{\Delta OFC} = \sin^{-1} \left\{ \frac{CF}{\sqrt{(CF)^2 + (OF)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CF}{\sqrt{(CF)^2 + (OF)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OF)^2}} \right) \right\}$$

$$= \sin^{-1} \left\{ \frac{2.1}{\sqrt{(2.1)^2 + (3.65)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{2.1}{\sqrt{(2.1)^2 + (3.65)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (3.65)^2}} \right) \right\}$$

$$= 0.522091613 - 0.37726383 = \mathbf{0.144827783 \text{ sr}}$$

$$\Rightarrow \omega_{\Delta OHC} = \sin^{-1} \left\{ \frac{CH}{\sqrt{(CH)^2 + (OH)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CH}{\sqrt{(CH)^2 + (OH)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OH)^2}} \right) \right\}$$

$$= \sin^{-1} \left\{ \frac{1.9}{\sqrt{(1.9)^2 + (3.8)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{1.9}{\sqrt{(1.9)^2 + (3.8)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (3.8)^2}} \right) \right\}$$

$$= 0.463647609 - 0.330197223 = \mathbf{0.133450386 \text{ sr}}$$

Hence, by setting the corresponding values in eq(II), solid angle subtended by the pentagonal plane at the given point 'P' is calculated as follows

$$\begin{aligned}\omega_{ABCD} &= \omega_{\Delta OEA} + \omega_{\Delta OGA} + \omega_{\Delta OFD} - \omega_{\Delta OEB} - \omega_{\Delta OHB} - \omega_{\Delta OGD} - \omega_{\Delta OFC} - \omega_{\Delta OHC} \\ &= 0.518779129 + 0.54915404 + 0.408539709 - 0.096487988 - 0.385912371 - 0.405557371 \\ &\quad - 0.144827783 - 0.133450386 = \mathbf{0.310236979 \text{ sr}}\end{aligned}$$

Calculation of Luminous Flux: If a **uniform point-source** of 1400 lm is located at the given point 'P' then the **total luminous flux intercepted** by the quadrilateral plane ABCD

$$= \frac{\text{solid angle} \times \text{total flux emitted by source}}{4\pi} = \frac{0.310236979 \times 1400}{4\pi} = \mathbf{34.56302412 \text{ lm (Lumen)}}$$

It means that only 34.56302412 lm out of 1400 lm flux is striking the quadrilateral plane ABCD & rest of the flux is escaping to the surrounding space. This result can be experimentally verified. (H.C. Rajpoot)

Thus, all the mathematical results obtained above can be verified by the experimental results. Although, there had not been any unifying principle to be applied on any polygonal plane for any configuration & location of the point in the space. The symbols & names used above are arbitrary given by the author Mr H.C. Rajpoot.

Note: This hand book has been written by the author in good faith that it will help to the learners to memorise the formulae from the book "Advanced Geometry". The derivations & detailed explanation have been given in the book.

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Published Papers of the author by International Journals of Mathematics

"HCR's Rank or Series Formula"	IJMPSR	March-April, 2014
"HCR's Series (Divergence)"	IOSR	March-April, 2014
"HCR's Infinite-series (Convergence)"	IJMPSR	Oct, 2014
"HCR's Theory of Polygon"	IJMPSR	Oct, 2014

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