$\boldsymbol{Mathematical\ Analysis\ of\ Spherical\ Triangle}$

$\boldsymbol{Application\ of\ HCR's\ Inverse\ Cosine\ Formula\ \&\ Theory\ of\ Polygon}$

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1. Introduction: We very well know that a **spherical triangle is a triangle having all its three vertices at the spherical surface & each of its sides as a great circle arc.** It mainly differs from a plane triangle by having the sum of all its interior angles greater than 180^o $(\boldsymbol{property\ of\ a\ spherical\ triangle})$. (See figure 1 below)

2. Analysis of spherical triangle (when all of its sides are known): Consider any spherical ΔABC having all its sides (each as a great circle arc) of lengths $a, b\ \&\ c$ $(\forall\ a \leq b \leq c)$ on a spherical surface with a radius R such that its interior angles are $A, B\ \&\ C$ $(\forall\ A + B + C > \pi)$ (as shown in the figure 1)

Interior angles $\boldsymbol{A}, \boldsymbol{B}\ \&\ \boldsymbol{C}$ of spherical triangle: We know that each interior angle of a spherical triangle is the angle between the planes of great circle arcs representing any two of its consecutive sides. Now, join the vertices A, B & C by straight lines to obtain a **corresponding plane $\Delta \boldsymbol{ABC}$** (as shown by the dotted lines AB, BC & CA). Similarly, we can extend the straight lines OA, OB & OC to obtain a plane$\Delta A'B'C'$ which is the base of tetrahedron OA'B'C'.

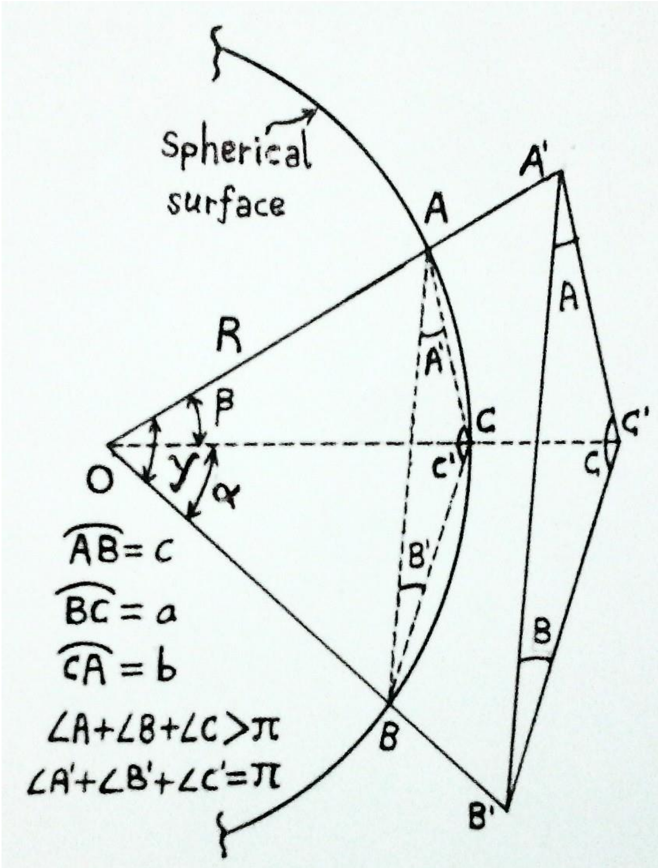

Figure 1: A spherical triangle ABC having its sides (each as a great circle arc) of lengths $\boldsymbol{a}, \boldsymbol{b}\ \&\ \boldsymbol{c}$ & its interior angles $\boldsymbol{A}$, $\boldsymbol{B}\ \&\ \boldsymbol{C}$. A plane $\Delta \boldsymbol{ABC}$ corresponding to the spherical triangle ABC is obtained by joining the vertices A, B & C by the straight lines.

Now, consider the **tetrahedron OA'B'C'** having angles $\alpha, \beta\ \&\ \gamma$ between its consecutive lateral edges OB' & OC', OA' & OC' and OA' & OB' respectively. Now the angles $\alpha, \beta\ \&\ \gamma$ are the angles subtended by the sides (each as a great circle arc) of spherical triangle at the centre of sphere which are determined as follows

$$\boldsymbol{\alpha = \frac{arc\ length}{radius} = \frac{a}{R}},\ \boldsymbol{\beta = \frac{b}{R}\ \&\ \gamma = \frac{c}{R}}$$

Now the interior angles A, $B\ \&\ C$ of spherical triangle that are also the angles between consecutive lateral triangular faces of the tetrahedron OA'B'C' meeting at the vertex O (i.e. the centre of sphere), are determined/calculated by using **HCR's Inverse Cosine Formula** according to which **if $\boldsymbol{\alpha}, \boldsymbol{\beta}\ \&\ \boldsymbol{\gamma}$ are the angles between consecutive lateral edges meeting at any of four vertices of a tetrahedron then the angle (opposite to $\boldsymbol{\alpha}$) between two lateral faces** is given as follows

$$\theta = cos^{-1}\left(\frac{cos\alpha - cos\beta cos\gamma}{sin\beta sin\gamma}\right)$$

$$\therefore\ \boldsymbol{A} = cos^{-1}\left(\frac{cos\alpha - cos\beta cos\gamma}{sin\beta sin\gamma}\right) = \boldsymbol{cos^{-1}\left(\frac{cos\frac{a}{R} - cos\frac{b}{R}cos\frac{c}{R}}{sin\frac{b}{R}sin\frac{c}{R}}\right)}$$

$$similarly, \quad \boldsymbol{B} = cos^{-1}\left(\frac{cos\beta - cos\alpha cos\gamma}{sin\alpha sin\gamma}\right) = \boldsymbol{cos^{-1}\left(\frac{cos\frac{b}{R} - cos\frac{a}{R}cos\frac{c}{R}}{sin\frac{a}{R}sin\frac{c}{R}}\right)}$$

$$\boldsymbol{C} = cos^{-1}\left(\frac{cos\gamma - cos\alpha cos\beta}{sin\alpha sin\beta}\right) = \boldsymbol{cos^{-1}\left(\frac{cos\frac{c}{R} - cos\frac{a}{R}cos\frac{b}{R}}{sin\frac{a}{R}sin\frac{b}{R}}\right)}$$

Area of spherical triangle: In order to calculate area covered by spherical triangle ABC, let's first calculate the solid angle subtended by it at the centre of sphere. But if we join the vertices A, B & C of the spherical triangle by straight lines then we obtain a **corresponding plane** $\boldsymbol{\Delta ABC}$ which exerts a solid angle equal to that subtended by the spherical triangle at the centre of sphere. Thus we would calculate the solid angle subtended by the corresponding plane ΔABC at the centre of sphere by two methods 1) **Analytic** & 2) **Graphical** as given below.

1. Analytic method for calculation of solid angle:

Sides of corresponding plane$\boldsymbol{\Delta ABC}$**:** Let the sides of corresponding plane ΔABC be $\boldsymbol{a', b' \& c'}$ ***opposite to its angles*** $\boldsymbol{A', \; B' \; \& \; C'} \; (\forall \; \boldsymbol{A' + B' + C' = \pi})$

In isosceles ΔOBC

$$\Rightarrow \sin\frac{\angle BOC}{2} = \frac{\left(\frac{BC}{2}\right)}{OB} \quad \Rightarrow sin\frac{\alpha}{2} = \frac{\left(\frac{a'}{2}\right)}{R} \quad \Rightarrow \quad a' = 2Rsin\frac{\alpha}{2} = 2Rsin\frac{a}{2R} \quad \left(since, \;\; \alpha = \frac{a}{R}\right)$$

$$\boldsymbol{a' = 2Rsin\frac{a}{2R}}$$

$$similarly, \quad \boldsymbol{b' = 2Rsin\frac{b}{2R}} \quad \& \quad \boldsymbol{c' = 2Rsin\frac{c}{2R}}$$

Now from **HCR's Axiom-2,** we know that **the perpendicular drawn from the centre of the sphere always passes through circumscribed centre of the plane triangle (in this case plane $\boldsymbol{\Delta ABC}$) obtained by joining the vertices of a spherical triangle to the centre of sphere** (See the figure 2)

Hence, the **circumscribed radius ($\boldsymbol{R'}$) of plane $\boldsymbol{\Delta ABC}$** having its sides $a', b' \& c' \,(all \; known)$ is given as follows

$$\boldsymbol{R' = \frac{a'b'c'}{4\Delta}}$$

Where,

$$\boldsymbol{Area \; of \; plane \; \Delta ABC, \;\; \Delta = \sqrt{s(s-a')(s-b')(s-c')}}$$

$$\boldsymbol{s = \frac{a'+b'+c'}{2}}$$

Hence, the **normal height ($\boldsymbol{h}$) of plane $\boldsymbol{\Delta ABC}$ from the centre O of the sphere** is given as follows

In right $\Delta OO'A$

$$OO' = \sqrt{(OA)^2 - (AO')^2}$$

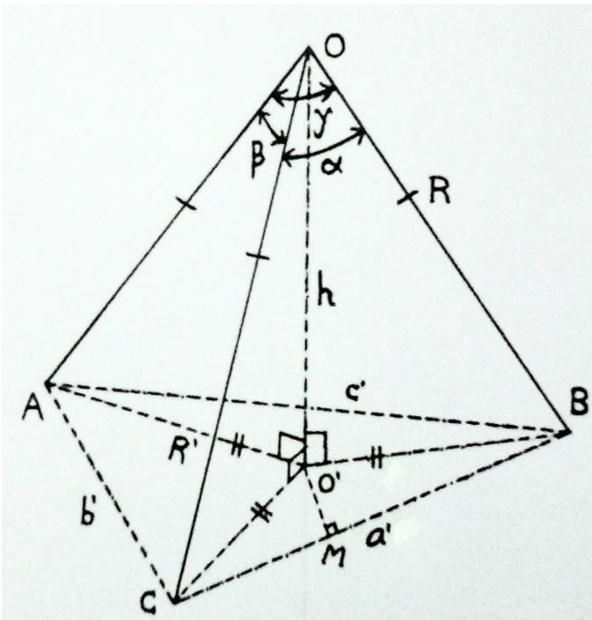

Figure 2: The perpendicular OO' drawn from the centre O of the sphere to the plane $\boldsymbol{\Delta ABC}$ always passes through its circumscribed centre O' according to HCR Axiom-2

$$\therefore\ \boldsymbol{h = \sqrt{R^2 - R'^2}}$$

Now, in right $\Delta \boldsymbol{O'MB}$

$$\boldsymbol{O'M} = \sqrt{(BO')^2 - (MB)^2} = \sqrt{R'^2 - \left(\frac{a'}{2}\right)^2} = \boldsymbol{\frac{\sqrt{4R'^2 - a'^2}}{2}} \qquad \left(since,\ \ CM = MB = \frac{a'}{2}\right)$$

Now, from **HCR's Theory of Polygon, the solid angle subtended by the right triangle having its orthogonal sides $\boldsymbol{a}$ & $\boldsymbol{b}$ at any point lying at a height $\boldsymbol{h}$ on the vertical axis passing through the vertex common to the side $\boldsymbol{a}$ & the hypotenuse** is given from standard formula as

$$\boldsymbol{\omega = \sin^{-1}\left(\frac{b}{\sqrt{b^2 + a^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2 + a^2}}\right)\left(\frac{h}{\sqrt{h^2 + a^2}}\right)\right\}}$$

Hence, the **solid angle $(\boldsymbol{\omega_{\Delta O'BC}})$ subtended by the isosceles $\Delta \boldsymbol{O'BC}$ at the centre O of the sphere**

$$= \omega_{\Delta O'MB} + \omega_{\Delta O'MC} = 2(\omega_{\Delta O'MB}) = 2(solid\ angle\ subtended\ by\ the\ right\ \Delta O'MB)$$

Hence, by setting the corresponding values in the above formula, we get

$$\omega_{\Delta O'BC} = 2\left[\sin^{-1}\left(\frac{\frac{a'}{2}}{\sqrt{\left(\frac{a'}{2}\right)^2 + \left(\frac{\sqrt{4R'^2 - a'^2}}{2}\right)^2}}\right) - \sin^{-1}\left\{\left(\frac{\frac{a'}{2}}{\sqrt{\left(\frac{a'}{2}\right)^2 + \left(\frac{\sqrt{4R'^2 - a'^2}}{2}\right)^2}}\right)\left(\frac{\sqrt{R^2 - R'^2}}{\sqrt{\left(\sqrt{R^2 - R'^2}\right)^2 + \left(\frac{\sqrt{4R'^2 - a'^2}}{2}\right)^2}}\right)\right\}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{a'}{2\sqrt{\frac{a'^2}{4} + R'^2 - \frac{a'^2}{4}}}\right) - \sin^{-1}\left\{\left(\frac{a'}{2\sqrt{\frac{a'^2}{4} + R'^2 - \frac{a'^2}{4}}}\right)\left(\frac{\sqrt{R^2 - R'^2}}{\sqrt{R^2 - \frac{1}{4}a'^2}}\right)\right\}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{a'}{2R'}\right) - \sin^{-1}\left\{\left(\frac{a'}{2R'}\right)\left(\frac{\sqrt{R^2 - R'^2}}{\sqrt{R^2 - \frac{1}{4}\left(2R\sin\frac{a}{2R}\right)^2}}\right)\right\}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{a'}{2R'}\right) - \sin^{-1}\left\{\left(\frac{a'}{2R'}\right)\left(\frac{\sqrt{R^2 - R'^2}}{R\cos\frac{a}{2R}}\right)\right\}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{a'}{2R'}\right) - \sin^{-1}\left(\left(\frac{a'}{2R'}\right)sec\frac{a}{2R}\sqrt{1 - \left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\boldsymbol{\omega_{\Delta O'BC} = 2\left[\sin^{-1}\left(\frac{a'}{2R'}\right) - \sin^{-1}\left(\left(\frac{a'}{2R'}\right)sec\frac{a}{2R}\sqrt{1 - \left(\frac{R'}{R}\right)^2}\right)\right] = \omega_1\ (let)}$$

Similarly, we have

$$\omega_{\Delta O'AC} = 2\left[\sin^{-1}\left(\frac{b'}{2R'}\right) - \sin^{-1}\left(\left(\frac{b'}{2R'}\right)sec\frac{b}{2R}\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right] = \omega_2 \ \ (let)$$

$$\omega_{\Delta O'AB} = 2\left[\sin^{-1}\left(\frac{c'}{2R'}\right) - \sin^{-1}\left(\left(\frac{c'}{2R'}\right)sec\frac{c}{2R}\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right] = \omega_3 \ \ (let)$$

Now, we must check out the nature of plane ΔABC whether it is an acute, a right or an obtuse triangle. Since the largest side is c' $among$ a' & b' hence we can determine the largest angle C' of plane ΔABC using cosine formula as follows

$$cosC' = \frac{a'^2 + b'^2 - c'^2}{2a'b'}$$

Thus, there arise two cases to calculate the solid angle subtended by the plane ΔABC at the centre of sphere & so by the spherical triangle ABC as follows

Case 1: Corresponding plane ΔABC is an acute or a right triangle ($\forall\ c' \geq b' \geq a'$ & $C' \leq 90^o$):

In this case, the foot point O' of the perpendicular drawn from the centre of sphere to the acute plane ΔABC lies within or on the boundary of this triangle. **All the values of solid angles ω_1, ω_2 & ω_3 corresponding to all the sides a', b' & c' respectively of acute plane ΔABC are taken as positive**. Hence, the solid angle $(\omega_{\Delta ABC})$ subtended by the acute plane ΔABC at the centre of sphere is given as the **sum of magnitudes of solid angles** as follows

$$\omega = \omega_{\Delta ABC} = \omega_{\Delta O'BC} + \omega_{\Delta O'AC} + \omega_{\Delta O'AB} = \omega_1 + \omega_2 + \omega_3$$

$$\therefore\ Area\ covered\ by\ the\ spherical\ triangle\ ABC = \omega R^2 = R^2(\omega_1 + \omega_2 + \omega_3)$$

Case 2: Corresponding plane ΔABC is an obtuse triangle ($\forall\ c' > b' \geq a'$ & $C' > 90^o$):

In this case, the foot point O' of the perpendicular drawn from the centre of sphere to the obtuse plane ΔABC lies outside the boundary of this triangle. (See the figure 3 below). In this case, **solid angles ω_1 & ω_2 corresponding to the sides a' & b' respectively are taken as positive while solid angle ω_3 corresponding to the largest side c' of obtuse plane ΔABC is taken as negative**. Hence, the solid angle $(\omega_{\Delta ABC})$ subtended by the obtuse plane ΔABC at the centre of sphere is given as the **algebraic sum of solid angles** as follows

$$\omega = \omega_{\Delta ABC} = \omega_{\Delta O'BC} + \omega_{\Delta O'AC} - \omega_{\Delta O'AB} = \omega_1 + \omega_2 - \omega_3$$

$$\therefore\ Area\ covered\ by\ the\ spherical\ triangle\ ABC = \omega R^2 = R^2(\omega_1 + \omega_2 - \omega_3)$$

2. Graphical method for calculation of solid angle:

In this method, we first plot the diagram of corresponding plane ΔABC having known sides a', b' & c' & then specify the location of **foot of perpendicular (F.O.P.)** i.e. the **circumscribed centre** of plane ΔABC then draw the **perpendiculars from circumscribed centre to all the opposite sides to divide it (i.e. plane ΔABC) into elementary right triangles** & use **standard formula-1 of right triangle** for calculating the solid angle subtended by each of the elementary right triangles at the centre of sphere which is given as follows

$$\boldsymbol{\omega = \sin^{-1}\left(\frac{b}{\sqrt{b^2 + a^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2 + a^2}}\right)\left(\frac{h}{\sqrt{h^2 + a^2}}\right)\right\}}$$

Then find out the algebraic sum (ω) of the solid angles subtended by the elementary right triangles at the centre of the sphere & hence the area covered by the spherical triangle ABC

$$\boldsymbol{Area\ covered\ by\ the\ spherical\ triangle\ ABC = \omega R^2}$$

3. Analysis of spherical triangle (when two of its sides & an interior angle between them are known): Consider any spherical triangle ΔABC, having its two sides (each as a great circle arc) of lengths $\boldsymbol{a}$ & $\boldsymbol{b}$ and an interior angle $\boldsymbol{C}$ between them, on a spherical surface with a radius $\boldsymbol{R}$. Now we can easily determine all its **unknown parameters i.e. unknown side (c), two interior angles A & B and area covered by it.**

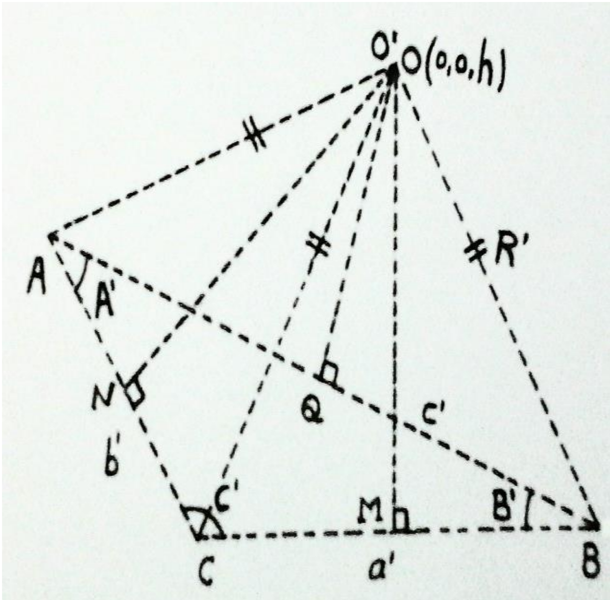

Figure 3: Corresponding plane ΔABC is an obtuse triangle $\forall\ c' > b' \geq a'$ & $C' > 90^o$. Centre O $(0, 0, h)$ of the sphere is lying at a height h perpendicular outwards to the plane of paper

Now the angles α, β & γ are the angles subtended by the sides (each as a great circle arc) of spherical triangle at the centre of sphere which are determined as follows (See the figure 2 above)

$$\boldsymbol{\alpha = \frac{arc\ length}{radius} = \frac{a}{R}\ ,\ \beta = \frac{b}{R}\ \ \&\ \ \gamma = \frac{c}{R} = ?\ (since,\ \ c = ?)}$$

Now, apply **HCR's Inverse cosine formula** for known interior angle C as follows

$$C = cos^{-1}\left(\frac{cos\gamma - cos\alpha cos\beta}{sin\alpha sin\beta}\right) = cos^{-1}\left(\frac{cos\frac{c}{R} - cos\frac{a}{R}cos\frac{b}{R}}{sin\frac{a}{R}sin\frac{b}{R}}\right)$$

$$\Rightarrow\ \frac{cos\frac{c}{R} - cos\frac{a}{R}cos\frac{b}{R}}{sin\frac{a}{R}sin\frac{b}{R}} = cosC\ \Rightarrow\ cos\frac{c}{R} = sin\frac{a}{R}sin\frac{b}{R}cosC + cos\frac{a}{R}cos\frac{b}{R}$$

$$\therefore\ \boldsymbol{c = R\cos^{-1}\left(sin\frac{a}{R}sin\frac{b}{R}cosC + cos\frac{a}{R}cos\frac{b}{R}\right)\ \ \&\ \ \gamma = \frac{c}{R} = \cos^{-1}\left(sin\frac{a}{R}sin\frac{b}{R}cosC + cos\frac{a}{R}cos\frac{b}{R}\right)}$$

Again by applying HCR's Inverse cosine formula for calculating the unknown interior angle A & B as follows

$$\boldsymbol{A} = cos^{-1}\left(\frac{cos\alpha - cos\beta cos\gamma}{sin\beta sin\gamma}\right) = \boldsymbol{cos^{-1}\left(\frac{cos\frac{a}{R} - cos\frac{b}{R}cos\frac{c}{R}}{sin\frac{b}{R}sin\frac{c}{R}}\right)}$$

$$\boldsymbol{B} = cos^{-1}\left(\frac{cos\beta - cos\alpha cos\gamma}{sin\alpha sin\gamma}\right) = \boldsymbol{cos^{-1}\left(\frac{cos\frac{b}{R} - cos\frac{a}{R}cos\frac{c}{R}}{sin\frac{a}{R}sin\frac{c}{R}}\right)}$$

Area of spherical triangle: In order to calculate area covered by the spherical triangle ABC, let's first calculate the solid angle subtended by it at the centre of sphere. But if we join the vertices A, B & C of spherical triangle by the straight lines then we obtain a **corresponding plane ΔABC** which exerts a solid angle equal to that subtended by the spherical triangle ABC at the centre of sphere. Now all the sides a', b' & c' of the plane ΔABC can be calculated by following the previous method (as mentioned above) as follows

$$\boldsymbol{a' = 2R\sin\frac{a}{2R}, \quad b' = 2R\sin\frac{b}{2R} \ \& \ c' = 2R\sin\frac{c}{2R}}$$

Thus we can calculate the solid angle subtended by the corresponding plane ΔABC & so by the spherical triangle ABC at the centre of sphere by following the previous two methods 1) **Analytic** & 2) **Graphical** (See the above procedures). Hence we can calculate the area covered by the given spherical triangle.

Illustrative Numerical Examples

These examples are based on all above articles which are very practical and directly & simply applicable to calculate the different parameters of a spherical triangle. For **ease of understanding & the calculations**, the **value of side c of the spherical triangle ABC is taken as the largest one**).

Example 1: **Calculate the area & each of the interior angles of a spherical triangle, having its sides (each as a great circle arc) of lengths 12, 18 & 20 units, on the spherical surface with a radius 50 units.**

Sol. Here, we have

$$R = 50\,units, \ \ a = 12\,units, \ \ b = 18\,units, \ \ c = 20\,units \ \Rightarrow \ A, B, C = ? \ \& \ Area = ?$$

Now, all the interior angles of spherical triangle can be easily calculated by using inverse cosine formula as follows

$$\Rightarrow \ \boldsymbol{A} = cos^{-1}\left(\frac{cos\frac{a}{R} - cos\frac{b}{R}cos\frac{c}{R}}{sin\frac{b}{R}sin\frac{c}{R}}\right) = cos^{-1}\left(\frac{cos\frac{12}{50} - cos\frac{18}{50}cos\frac{20}{50}}{sin\frac{18}{50}sin\frac{20}{50}}\right) \approx \boldsymbol{37.165231^o \approx 37^o9'54.83''}$$

$$\boldsymbol{B} = cos^{-1}\left(\frac{cos\frac{b}{R} - cos\frac{a}{R}cos\frac{c}{R}}{sin\frac{a}{R}sin\frac{c}{R}}\right) = cos^{-1}\left(\frac{cos\frac{18}{50} - cos\frac{12}{50}cos\frac{20}{50}}{sin\frac{12}{50}sin\frac{20}{50}}\right) \approx \boldsymbol{63.54656423^o \approx 63^o32'47.63''}$$

$$\boldsymbol{C} = cos^{-1}\left(\frac{cos\frac{c}{R} - cos\frac{a}{R}cos\frac{b}{R}}{sin\frac{a}{R}sin\frac{b}{R}}\right) = cos^{-1}\left(\frac{cos\frac{20}{50} - cos\frac{12}{50}cos\frac{18}{50}}{sin\frac{12}{50}sin\frac{18}{50}}\right) \approx \boldsymbol{81.76846174^o \approx 81^o46'6.46''}$$

$$\Rightarrow \boldsymbol{A + B + C > 180^o} \quad (\boldsymbol{property\ of\ spherical\ triangle})$$

Now, the **sides of corresponding plane ΔABC** are calculated as follows

$$a' = 2Rsin\frac{a}{2R} = 2(50)sin\frac{12}{100} \approx 11.97122073$$

$$b' = 2Rsin\frac{b}{2R} = 2(50)sin\frac{18}{100} \approx 17.90295734$$

$$c' = 2Rsin\frac{c}{2R} = 2(50)sin\frac{20}{100} \approx 19.86693308$$

$$s = semiperimeter = \frac{a' + b' + c'}{2} \approx \frac{11.97122073 + 17.90295734 + 19.86693308}{2} \approx 24.87055558$$

Area of plane ΔABC is given as

$$\Delta = \sqrt{s(s-a')(s-b')(s-c')}$$

$$\approx \sqrt{24.87055558(24.87055558 - 11.97122073)(24.87055558 - 17.90295734)(24.87055558 - 19.86693308)}$$
$$\approx \sqrt{24.87055558 \times 12.89933485 \times 6.96759824 \times 5.0036225} \approx 105.7572673$$

$$\therefore \boldsymbol{circumscribed\ radius, R'} = \frac{a'b'c'}{4\Delta} \approx \frac{11.97122073 \times 17.90295734 \times 19.86693308}{4 \times 105.7572673} \approx 10.06523299$$

Since, the largest side of plane ΔABC is $c' \approx 19.86693308$ hence the largest angle of the plane ΔABC is C' which is calculated by using cosine formula as follows

$$cosC' = \frac{a'^2 + b'^2 - c'^2}{2a'b'} \Rightarrow C' = \cos^{-1}\left(\frac{a'^2 + b'^2 - c'^2}{2a'b'}\right)$$

$$C' \approx \cos^{-1}\left(\frac{(11.97122073)^2 + (17.90295734)^2 - (19.86693308)^2}{2(11.97122073)(17.90295734)}\right) \approx 80.71882239^o < 90^o$$

Hence, the **plane $\Delta \boldsymbol{ABC}$ is an acute angled triangle**.

Note: If all the interior angles A, B & C of any spherical triangle are acute then definitely the corresponding plane ΔABC will also be an acute angled triangle. It is not required to check it out by calculating the largest angle C' of plane ΔABC. (As in above example 1, we need not calculate the largest angle C' to check out the nature of the plane ΔABC we can directly say on the basis of values of interior angles A, B & C of the spherical surface that the plane ΔABC is an acute if each of A, B & C is an acute angle)

Hence the **foot of perpendicular (F.O.P.) drawn from the centre of sphere to the plane $\Delta \boldsymbol{ABC}$ will lie within the boundary of plane $\Delta \boldsymbol{ABC}$** (See the figure 2 above) hence, the **solid angle subtended by it at the centre of sphere** is calculated as follows

$$\boldsymbol{\omega_1} = 2\left[\sin^{-1}\left(\frac{a'}{2R'}\right) - \sin^{-1}\left(\left(\frac{a'}{2R'}\right) sec\frac{a}{2R}\sqrt{1 - \left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\approx 2\left[\sin^{-1}\left(\frac{11.97122073}{2(10.06523299)}\right) - \sin^{-1}\left(\left(\frac{11.97122073}{2(10.06523299)}\right) sec\frac{12}{2(50)}\sqrt{1 - \left(\frac{10.06523299}{50}\right)^2}\right)\right]$$
$$\approx \boldsymbol{0.019716827\ sr}$$

$$\boldsymbol{\omega_2} = 2\left[\sin^{-1}\left(\frac{b'}{2R'}\right) - \sin^{-1}\left(\left(\frac{b'}{2R'}\right) sec\frac{b}{2R}\sqrt{1 - \left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\approx 2\left[\sin^{-1}\left(\frac{17.90295734}{2(10.06523299)}\right) - \sin^{-1}\left(\left(\frac{17.90295734}{2(10.06523299)}\right) sec\frac{18}{2(50)}\sqrt{1 - \left(\frac{10.06523299}{50}\right)^2}\right)\right]$$
$$\approx \boldsymbol{0.016922497\ sr}$$

$$\boldsymbol{\omega_3} = 2\left[\sin^{-1}\left(\frac{c'}{2R'}\right) - \sin^{-1}\left(\left(\frac{c'}{2R'}\right) sec\frac{c}{2R}\sqrt{1 - \left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\approx 2\left[\sin^{-1}\left(\frac{19.86693308}{2(10.06523299)}\right)-\sin^{-1}\left(\left(\frac{19.86693308}{2(10.06523299)}\right)sec\frac{20}{2(50)}\sqrt{1-\left(\frac{10.06523299}{50}\right)^2}\right)\right]$$
$$\approx \mathbf{0.00664932472\ sr}$$

Note: In this case, **all the values of solid angles $\boldsymbol{\omega_1}$, $\boldsymbol{\omega_2}$ & $\boldsymbol{\omega_3}$ corresponding to all the sides $\boldsymbol{a', b'}$ & $\boldsymbol{c'}$ respectively of the acute plane ΔABC are taken as positive.**

Hence, the **solid angle $(\omega_{\Delta ABC})$ subtended by the acute plane ΔABC or spherical triangle ABC** at the centre of sphere is given as the **sum of magnitudes of solid angles** as follows

$$\boldsymbol{\omega}=\omega_1+\omega_2+\omega_3\approx 0.019716827+0.016922497\ +0.00664932472\approx \mathbf{0.043288648\ sr}$$

$$\therefore \boldsymbol{Area\ covered\ by\ the\ spherical\ triangle\ ABC=\omega R^2}\approx 0.043288648\times 50^2$$
$$\approx \mathbf{108.2216218\ unit^2} \qquad \boldsymbol{Ans}$$

The above value of area implies that the given **spherical triangle** covers $\approx \mathbf{108.2216218\ unit^2}$ of the total surface area $=\mathbf{4\pi(50)^2}\approx \mathbf{31415.92654\ unit^2}$ & subtends a **solid angle** $\approx \mathbf{0.043288648\ sr}$ at the centre of the sphere with a radius 50 units**.**

Example 2: **A spherical triangle, having its two sides (each as a great circle arc) of lengths 25 & 38 units and an interior angle $\mathbf{160^o}$ included by them, on the spherical surface with a radius 200 units. Calculate the unknown side, interior angles & the area covered by it.**

Sol. Here, we have

$$R=200\ units,\quad a=25\ units,\quad b=38\ units,\quad C=160^o=\frac{8\pi}{9}\ \Rightarrow c=?,\ \ A,B\ =?\ \&\ Area=?$$

Now in order to calculate unknown side c, apply **HCR's Inverse cosine formula** for known interior angle C as follows

$$C=cos^{-1}\left(\frac{cos\frac{c}{R}-cos\frac{a}{R}cos\frac{b}{R}}{sin\frac{a}{R}sin\frac{b}{R}}\right)\Rightarrow c=R\cos^{-1}\left(sin\frac{a}{R}sin\frac{b}{R}cosC+cos\frac{a}{R}cos\frac{b}{R}\right)$$

$$\boldsymbol{c}=200\cos^{-1}\left(sin\frac{25}{200}sin\frac{38}{200}cos\frac{8\pi}{9}+cos\frac{25}{200}cos\frac{38}{200}\right)\approx \mathbf{62.07679003}$$

Again by applying HCR's Inverse cosine formula for calculating the unknown interior angle A & B as follows

$$\boldsymbol{A}=cos^{-1}\left(\frac{cos\frac{a}{R}-cos\frac{b}{R}cos\frac{c}{R}}{sin\frac{b}{R}sin\frac{c}{R}}\right)=cos^{-1}\left(\frac{cos\frac{25}{200}-cos\frac{38}{200}cos\frac{62.07679003}{200}}{sin\frac{38}{200}sin\frac{62.07679003}{200}}\right)\approx \mathbf{8^o1'31.68''}$$

$$\boldsymbol{B}=cos^{-1}\left(\frac{cos\frac{b}{R}-cos\frac{a}{R}cos\frac{c}{R}}{sin\frac{a}{R}sin\frac{c}{R}}\right)=cos^{-1}\left(\frac{cos\frac{38}{200}-cos\frac{25}{200}cos\frac{62.07679003}{200}}{sin\frac{25}{200}sin\frac{62.07679003}{200}}\right)\approx \mathbf{12^o12'34.43''}$$

$$\Rightarrow \boldsymbol{A+B+C>180^o}\quad (\boldsymbol{property\ of\ spherical\ triangle})$$

Now, the **sides of corresponding plane ΔABC** are calculated as follows

$$a' = 2Rsin\frac{a}{2R} = 2(200)sin\frac{25}{400} \approx 24.98372714$$

$$b' = 2Rsin\frac{b}{2R} = 2(200)sin\frac{38}{400} \approx 37.94286745$$

$$c' = 2Rsin\frac{c}{2R} = 2(200)sin\frac{62.07679003}{400} \approx 61.82790801$$

$$s = semiperimeter = \frac{a'+b'+c'}{2} \approx \frac{24.98372714 + 37.94286745 + 61.82790801}{2} \approx 62.3772513$$

Area of plane ΔABC is given as

$$\Delta = \sqrt{s(s-a')(s-b')(s-c')}$$

$$\approx \sqrt{62.3772513(62.3772513 - 24.98372714)(62.3772513 - 37.94286745)(62.3772513 - 61.82790801)}$$
$$\approx \sqrt{62.3772513 \times 37.39352416 \times 24.43438385 \times 0.54934329} \approx 176.9432188$$

$$\therefore \boldsymbol{circumscribed\ radius, R'} = \frac{a'b'c'}{4\Delta} \approx \frac{24.98372714 \times 37.94286745 \times 61.82790801}{4 \times 176.9432188} \approx 82.80909039$$

Since, the largest side of plane ΔABC is $c' \approx 61.82790801$ hence the largest angle of the plane ΔABC is C' which is calculated by using cosine formula as follows

$$cosC' = \frac{a'^2 + b'^2 - c'^2}{2a'b'} \Rightarrow C' = \cos^{-1}\left(\frac{a'^2 + b'^2 - c'^2}{2a'b'}\right)$$

$$C' \approx \cos^{-1}\left(\frac{(24.98372714)^2 + (37.94286745)^2 - (61.82790801)^2}{2(24.98372714)(37.94286745)}\right) \approx 158.0797337^o > 90^o$$

Hence, the **plane ΔABC is an obtuse angled triangle**.

Hence the **foot of perpendicular (F.O.P.) drawn from the centre of sphere to the plane ΔABC will lie outside the boundary of plane ΔABC** (See the figure 3 above) hence, the **solid angle subtended by it at the centre of sphere** is calculated as follows

$$\boldsymbol{\omega_1} = 2\left[\sin^{-1}\left(\frac{a'}{2R'}\right) - \sin^{-1}\left(\left(\frac{a'}{2R'}\right)sec\frac{a}{2R}\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\approx 2\left[\sin^{-1}\left(\frac{24.98372714}{2(82.80909039)}\right) - \sin^{-1}\left(\left(\frac{24.98372714}{2(82.80909039)}\right)sec\frac{25}{2(200)}\sqrt{1-\left(\frac{82.80909039}{200}\right)^2}\right)\right]$$
$$\approx \boldsymbol{0.026819267\ sr}$$

$$\boldsymbol{\omega_2} = 2\left[\sin^{-1}\left(\frac{b'}{2R'}\right) - \sin^{-1}\left(\left(\frac{b'}{2R'}\right)sec\frac{b}{2R}\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\approx 2\left[\sin^{-1}\left(\frac{37.94286745}{2(82.80909039)}\right)-\sin^{-1}\left(\left(\frac{37.94286745}{2(82.80909039)}\right)sec\frac{38}{2(200)}\sqrt{1-\left(\frac{82.80909039}{200}\right)^2}\right)\right]$$

$$\approx \mathbf{0.04021067}\ \boldsymbol{sr}$$

$$\boldsymbol{\omega_3}=2\left[\sin^{-1}\left(\frac{c'}{2R'}\right)-\sin^{-1}\left(\left(\frac{c'}{2R'}\right)sec\frac{c}{2R}\sqrt{1-\left(\frac{R'}{R}\right)^2}\right)\right]$$

$$\approx 2\left[\sin^{-1}\left(\frac{61.82790801}{2(82.80909039)}\right)-\sin^{-1}\left(\left(\frac{61.82790801}{2(82.80909039)}\right)sec\frac{62.07679003}{2(200)}\sqrt{1-\left(\frac{82.80909039}{200}\right)^2}\right)\right]$$

$$\approx \mathbf{0.062927892}\ \boldsymbol{sr}$$

Note: In this case, **solid angles $\boldsymbol{\omega_1}$ & $\boldsymbol{\omega_2}$ corresponding to the sides $\boldsymbol{a'}$ & $\boldsymbol{b'}$ respectively are taken as positive while solid angle $\boldsymbol{\omega_3}$ corresponding to the largest side $\boldsymbol{c'}$ of obtuse plane $\Delta \boldsymbol{ABC}$ is taken as negative.**

Hence, the **solid angle $(\boldsymbol{\omega_{\Delta ABC}})$ subtended by the obtuse plane $\Delta \boldsymbol{ABC}$ or spherical triangle ABC** at the centre of sphere is given as the **algebraic sum of solid angles** as follows

$$\boldsymbol{\omega}=\omega_1+\omega_2-\omega_3\approx \mathbf{0.026819267}+\mathbf{0.04021067}-\mathbf{0.062927892}\approx \mathbf{0.004102045}\ \boldsymbol{sr}$$

$$\therefore \boldsymbol{Area\ covered\ by\ the\ spherical\ triangle\ ABC}=\boldsymbol{\omega R^2}\approx 0.00646329\times 200^2$$

$$\approx \mathbf{164.0818}\ \boldsymbol{unit^2} \qquad \boldsymbol{Ans}$$

The above value of area implies that the given **spherical triangle** covers $\approx \mathbf{164.0818}\ \boldsymbol{unit^2}$ of the total surface area $=\mathbf{4\pi(200)^2}\approx \mathbf{502654.8246}\ \boldsymbol{unit^2}$ & subtends a **solid angle** $\approx \mathbf{0.004102045}\ \boldsymbol{sr}$ at the centre of the sphere with a radius 200 units**.**

Conclusion: *All the articles above have been derived by **Mr H.C. Rajpoot** by using **simple geometry & trigonometry**. All above articles (formula) are very practical & simple to apply in case of **a spherical triangle** to calculate all its important parameters such as solid angle, surface area covered, interior angles etc. & also useful for calculating all the parameters of the **corresponding plane triangle** obtained by joining all the vertices of a spherical triangle by the straight lines. These formulae can also be used to calculate all the parameters of the right pyramid obtained by joining all the vertices of a spherical triangle to the centre of sphere such as normal height, angle between the consecutive lateral edges, area of plane triangular base etc.*

Note: Above articles had been derived *& illustrated by **Mr H.C. Rajpoot** (**B Tech, Mechanical Engineering**)*

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