Mathematical Analysis of Spherical Rectangle

Application of HCR's Inverse Cosine Formula & Theory of Polygon

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1. Introduction: We know that a **spherical rectangle is a 3-D figure, on a spherical surface, enclosed by four sides (each as a great circle arc) such that each pair of the non-parallel & opposite sides is equal in arc length & all four interior angles are equal in magnitude but each is greater than 90^o (property of a spherical rectangle).** (See figure 1 below)

2. Analysis of spherical rectangle (when the length & the width are known): Consider any **spherical rectangle ABCD** having its length & width (each as a great circle arc) as l & b $(\forall\ l \geq b)$ respectively on a spherical surface with a radius R such that each interior angle is θ $(\forall\ \theta > 90^o)$ (as shown in the figure 1)

Interior angle (θ) of spherical rectangle: We know that each **interior angle of a spherical rectangle** is the **angle between the planes of great circle arcs representing any two of its consecutive sides**. Now, join all the vertices A, B, C & D by the straight lines to obtain a **corresponding plane rectangle ABCD** (as shown by the dotted lines AB, BC, CD & DA) having centre at the point O'. Now the length l' & the width b' of the plane rectangle ABCD are calculated as follows

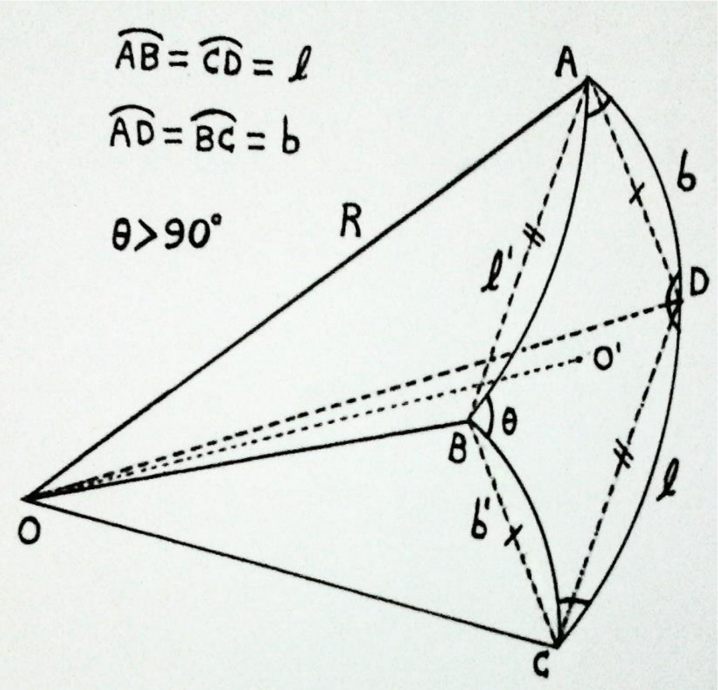

Figure 1: A spherical rectangle ABCD having its length & width (each as a great circle arc) as l & b respectively & each interior angle θ $(\forall\ \theta > 90^o)$. A plane rectangle ABCD corresponding to the spherical rectangle ABCD is obtained by joining the vertices A, B, C & D by the straight lines.

$$\angle AOB = \frac{arc\ length}{radius} = \frac{l}{R} \quad \& \quad \angle BOC = \frac{b}{R}$$

In (plane) isosceles ΔAOB

$$sin\left(\frac{\angle AOB}{2}\right) = \frac{\left(\frac{AB}{2}\right)}{OA}$$

$$sin\frac{l}{2R} = \frac{\frac{l'}{2}}{R} = \frac{l'}{2R}$$

$$l' = 2Rsin\frac{l}{2R} \quad \&\ similarly,\quad b' = 2Rsin\frac{b}{2R}$$

$$\therefore diagonal\ of\ the\ plane\ rectangle\ ABCD,\quad AC = \sqrt{l'^2 + b'^2}$$

$$\Rightarrow AC = \sqrt{\left(2Rsin\frac{l}{2R}\right)^2 + \left(2Rsin\frac{b}{2R}\right)^2} = 2R\sqrt{sin^2\frac{l}{2R} + sin^2\frac{b}{2R}} \quad \dots\dots\dots\dots\dots (I)$$

In (plane) isosceles ΔAOC

$$sin\left(\frac{\angle AOC}{2}\right)=\frac{\left(\frac{AC}{2}\right)}{OA}\ \Rightarrow\ sin\frac{acr\ AC}{2R}=\frac{AC}{2R}\ \left(\angle \boldsymbol{AOC}=\frac{\boldsymbol{arc\ AC}}{\boldsymbol{radius}}=\frac{\boldsymbol{arc\ AC}}{\boldsymbol{R}}\right)$$

$$\Rightarrow\frac{arc\ AC}{2R}=\sin^{-1}\left(\frac{AC}{2R}\right)\quad Or\quad arc\ AC=2R\sin^{-1}\left(\frac{2R\sqrt{sin^2\frac{l}{2R}+sin^2\frac{b}{2R}}}{2R}\right)\quad (from\ eq(I))$$

$$\therefore\ \boldsymbol{arc\ AC}=\boldsymbol{2R}\sin^{-1}\left(\sqrt{\boldsymbol{sin^2\frac{l}{2R}+sin^2\frac{b}{2R}}}\right)$$

Now, consider the **tetrahedron OABC** having angles $\angle \boldsymbol{AOB}, \angle \boldsymbol{BOC}$ & $\angle \boldsymbol{AOC}$ between its consecutive lateral edges OA & OB, OB & OC and OA & OC respectively meeting at the vertex O (i.e. centre of the sphere). Now these are the angles subtended by the sides (each as a great circle arc) AB, BC & CA respectively of the spherical triangle ABC at the centre of sphere which are determined as follows

$$\angle \boldsymbol{AOB}=\frac{\boldsymbol{arc\ length}}{\boldsymbol{radius}}=\frac{\boldsymbol{l}}{\boldsymbol{R}},\ \angle \boldsymbol{BOC}=\frac{\boldsymbol{b}}{\boldsymbol{R}}\ \ \&\ \ \angle \boldsymbol{AOC}=\frac{\boldsymbol{arc\ AC}}{\boldsymbol{R}}$$

Now **each of the interior angles $\boldsymbol{\theta}$ of the spherical rectangle ABCD is equal to the angle between consecutive lateral triangular faces $\Delta \boldsymbol{AOB}$ & $\Delta \boldsymbol{BOC}$ of the tetrahedron OABC meeting at the vertex O (i.e. the centre of sphere**), are determined/calculated by using **HCR's Inverse Cosine Formula** according to which **if $\boldsymbol{\alpha},\boldsymbol{\beta}$ & $\boldsymbol{\gamma}$ are the angles between consecutive lateral edges meeting at any of the four vertices of a tetrahedron then the angle (opposite to $\boldsymbol{\alpha}$) between two lateral faces** is given as follows

$$\theta=cos^{-1}\left(\frac{cos\alpha-cos\beta cos\gamma}{sin\beta sin\gamma}\right)$$

$$\therefore\ \boldsymbol{\theta}=cos^{-1}\left(\frac{cos\frac{arc\ AC}{R}-cos\frac{l}{R}cos\frac{b}{R}}{sin\frac{l}{R}sin\frac{b}{R}}\right)$$

$$=cos^{-1}\left(\frac{cos\left(\frac{2R\sin^{-1}\left(\sqrt{sin^2\frac{l}{2R}+sin^2\frac{b}{2R}}\right)}{R}\right)-cos\frac{l}{R}cos\frac{b}{R}}{sin\frac{l}{R}sin\frac{b}{R}}\right)$$

$$\boldsymbol{\theta}=\boldsymbol{cos^{-1}}\left(\frac{\boldsymbol{cos}\left(\boldsymbol{2}\sin^{-1}\left(\sqrt{\boldsymbol{sin^2\frac{l}{2R}+sin^2\frac{b}{2R}}}\right)\right)-\boldsymbol{cos\frac{l}{R}cos\frac{b}{R}}}{\boldsymbol{sin\frac{l}{R}sin\frac{b}{R}}}\right)$$

$$\forall\ \boldsymbol{l,b,R>0}\ \ \&\ \ \boldsymbol{l+b<\pi R}\ \ \Rightarrow\ \ \boldsymbol{90^o<\theta<180^o}$$

Above is the **required expression to determine each of the interior angles θ of any spherical rectangle having length l & width b (each as a great circle arc) on a spherical surface with a radius R.**

Area covered by the spherical rectangle: In order to calculate area covered by the spherical rectangle ABCD, let's first calculate the solid angle subtended by it at the centre O of the sphere. But if we join the all the vertices A, B, C & D of the spherical rectangle ABCD by the straight lines then we obtain a **corresponding plane rectangle ABCD** which exerts a solid angle equal to that subtended by the spherical rectangle ABCD at the centre of sphere. So let's calculate the normal height $OO' = h$ of the plane rectangle ABCD with centre O' from the centre O of the sphere as follows

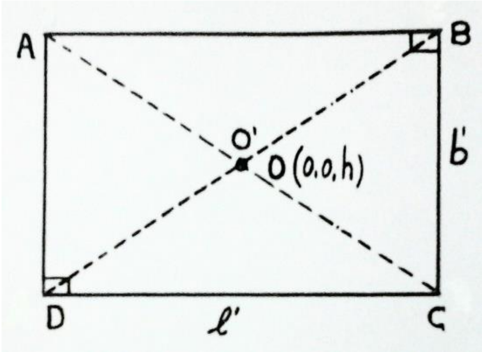

Figure 2: Plane rectangle ABCD is obtained by joining all the vertices A, B, C & D of spherical rectangle ABCD by the straight lines. The centre point O $(0, 0, h)$ is lying at a height h perpendicularly outwards to the plane of paper

In right $\Delta AO'O$ (perpendicular to the plane of paper)

$$OO' = \sqrt{(OA)^2 - (AO')^2} = \sqrt{(OA)^2 - \left(\frac{AC}{2}\right)^2} \quad \left(since,\ \ AO' = O'C = \frac{AC}{2}\right)$$

$$or\ h = \sqrt{(R)^2 - \left(\frac{2R\sqrt{sin^2\frac{l}{2R} + sin^2\frac{b}{2R}}}{2}\right)^2} \qquad (from\ eq(I))$$

$$= \sqrt{R^2 - R^2\left(sin^2\frac{l}{2R} + sin^2\frac{b}{2R}\right)} = R\sqrt{1 - sin^2\frac{l}{2R} - sin^2\frac{b}{2R}}$$

$$\boldsymbol{h = R\sqrt{cos^2\frac{b}{2R} - sin^2\frac{l}{2R}}} \qquad \dots\dots\dots\dots\dots\dots..(\boldsymbol{II})$$

Thus we calculate the solid angle (ω) subtended by the corresponding **plane rectangle ABCD** & so by the **spherical rectangle ABCD** at the centre of sphere by using **standard formula of rectangular plane** given by **HCR's Theory of Polygon** according to which the **solid angle (ω) subtended by any rectangular plane having length & width l** & **b respectively at any point lying at a normal height h from the centre** is given as

$$\omega = 4\sin^{-1}\left(\frac{lb}{\sqrt{(l^2 + 4h^2)(b^2 + 4h^2)}}\right)$$

Now by setting the corresponding values in the above expression, we get the solid angle subtended by the plane rectangle ABCD at the centre of sphere as follows

$$\omega = 4\sin^{-1}\left(\frac{l'b'}{\sqrt{\left(l'^2 + 4h^2\right)\left(b'^2 + 4h^2\right)}}\right)$$

$$= 4\sin^{-1}\left(\frac{\left(2Rsin\frac{l}{2R}\right)\left(2Rsin\frac{b}{2R}\right)}{\sqrt{\left(\left(2Rsin\frac{l}{2R}\right)^2 + 4\left(R\sqrt{cos^2\frac{b}{2R} - sin^2\frac{l}{2R}}\right)^2\right)\left(\left(2Rsin\frac{b}{2R}\right)^2 + 4\left(R\sqrt{cos^2\frac{b}{2R} - sin^2\frac{l}{2R}}\right)^2\right)}}\right)$$

$$= 4\sin^{-1}\left(\frac{4R^2 \sin\frac{l}{2R}\sin\frac{b}{2R}}{4R^2\sqrt{\left(\sin^2\frac{l}{2R}+\cos^2\frac{b}{2R}-\sin^2\frac{l}{2R}\right)\left(\sin^2\frac{b}{2R}+\cos^2\frac{b}{2R}-\sin^2\frac{l}{2R}\right)}}\right)$$

$$= 4\sin^{-1}\left(\frac{\sin\frac{l}{2R}\sin\frac{b}{2R}}{\sqrt{\left(\cos^2\frac{b}{2R}\right)\left(1-\sin^2\frac{l}{2R}\right)}}\right) = 4\sin^{-1}\left(\frac{\sin\frac{l}{2R}\sin\frac{b}{2R}}{\sqrt{\left(\cos^2\frac{b}{2R}\right)\left(\cos^2\frac{l}{2R}\right)}}\right) = 4\sin^{-1}\left(\frac{\sin\frac{l}{2R}\sin\frac{b}{2R}}{\cos\frac{l}{2R}\cos\frac{b}{2R}}\right)$$

$$\Rightarrow \boldsymbol{\omega = 4\sin^{-1}\left(\tan\frac{l}{2R}\tan\frac{b}{2R}\right)}$$

Above is the required expression for calculating the solid angle subtended by any spherical rectangle having length l & width b (each as a great circle arc) at the centre of a spherical surface with a radius R.

Hence, the area (A) covered by the spherical rectangle ABCD is given as

$$A = \omega \times (radius)^2 = \omega R^2 = 4\sin^{-1}\left(\tan\frac{l}{2R}\tan\frac{b}{2R}\right)\times R^2$$

$$\boldsymbol{A = 4R^2\sin^{-1}\left(\tan\frac{l}{2R}\tan\frac{b}{2R}\right) \quad \forall\ l, b, R > 0\ \&\ l + b < \pi R}$$

Above is the required expression for calculating the area covered by any spherical rectangle having length l & width b (each as a great circle arc) on a spherical surface with a radius R.

Illustrative Numerical Example

This example is based on all above articles which are very practical and directly & simply applicable to calculate the different parameters of any spherical rectangle such as the interior angle & the area covered by it.

Example 1: **Calculate the area & each of the interior angles of a spherical rectangle, having its length & width (each as a great circle arc) of 15 & 4 units respectively, on a spherical surface with a radius 40 units.**

Sol. Here, we have

$$R = 40\ units,\quad l = 15\ units,\quad b = 4\ units\ \Rightarrow\ \theta = ?\ \&\ Area\ (A) = ?$$

Now, each of the interior angles (θ) of the given spherical rectangle can be easily calculated by using formula as follows

$$\theta = \cos^{-1}\left(\frac{\cos\left(2\sin^{-1}\left(\sqrt{\sin^2\frac{l}{2R}+\sin^2\frac{b}{2R}}\right)\right)-\cos\frac{l}{R}\cos\frac{b}{R}}{\sin\frac{l}{R}\sin\frac{b}{R}}\right)$$

Now by setting the corresponding values of R, l & b, we get

$$\theta = \cos^{-1}\left(\frac{\cos\left(2\sin^{-1}\left(\sqrt{\sin^2\frac{15}{2(40)}+\sin^2\frac{4}{2(40)}}\right)\right)-\cos\frac{15}{40}\cos\frac{4}{40}}{\sin\frac{15}{40}\sin\frac{4}{40}}\right) \approx 90.543994^o \approx \mathbf{90^o 32' 38.38''}$$

$$\Rightarrow \boldsymbol{\theta > 90^o} \quad (\boldsymbol{property\ of\ a\ spherical\ rectangle})$$

Now, the area $(\boldsymbol{A})$ **covered by the spherical rectangle** on the spherical surface is given as follows

$$\boldsymbol{A} = 4R^2 \sin^{-1}\left(tan\frac{l}{2R}tan\frac{b}{2R}\right) = 4(40)^2 \sin^{-1}\left(tan\frac{15}{2(40)}tan\frac{4}{2(40)}\right)$$

$$= 6400 \sin^{-1}\left(tan\frac{15}{80}tan\frac{1}{20}\right) \approx \boldsymbol{60.76471331\ unit^2}$$

While the solid angle subtended by the spherical rectangle at the centre of sphere is given as follows

$$\boldsymbol{\omega} = 4 \sin^{-1}\left(tan\frac{l}{2R}tan\frac{b}{2R}\right) = 4 \sin^{-1}\left(tan\frac{15}{80}tan\frac{4}{80}\right) \approx \boldsymbol{0.037977945\ sr}$$

The above value of area implies that the given **spherical rectangle** covers $\approx \boldsymbol{60.76471331\ unit^2}$ of the total surface area $= \boldsymbol{4\pi(40)^2} \approx \boldsymbol{20106.19298\ unit^2}$& subtends a **solid angle** $\approx \boldsymbol{0.037977945\ sr}$ at the centre of the sphere with a radius 40 units**.**

Conclusion: *All the articles above have been derived by* ***Mr H.C. Rajpoot*** *by using* ***simple geometry & trigonometry****. All above articles (formula) are very practical & simple to apply in case of* ***any spherical rectangle*** *to calculate all its important parameters such as solid angle, surface area covered, interior angles etc. & also useful for calculating all the parameters of the* ***corresponding plane rectangle*** *obtained by joining all the vertices of a spherical rectangle by the straight lines. These formulae can also be used to calculate all the parameters of the right pyramid obtained by joining all the vertices of a spherical rectangle to the centre of the sphere such as normal height, angle between the consecutive lateral edges, area of plane rectangular base etc.*

Note: *Above articles had been derived & illustrated by* ***Mr H.C. Rajpoot (B Tech, Mechanical Engineering)***

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