Mathematical Analysis of Elliptical Path in the Annulus of Two Circles

Mr Harish Chandra Rajpoot   *Feb, 2015*

M.M.M. University of Technology, Gorakhpur-273010 (UP), India

1. Introduction:

When a smaller circle exactly lies inside the bigger one (either touching or not touching internally) with their centres separated by a certain distance, then the **locus of the centre of a third tangent (parametric) circle**, **in the annular region between them, touching the smaller circle externally & the bigger one internally is always an ellipse having the centres of former circles as its foci & its centre at the midpoint of the centres of former circles.** This path becomes a circle if both the smaller & bigger circles are concentric i.e. centres of both the circles, smaller inside bigger, are coincident. (See figure 1 below, showing an elliptical path (dotted curve A'BAB'A') in the annular region between two circles, smaller inside bigger, with their centres separated by a certain distance)

2. Derivation of major axis, minor axis, eccentricity & equation of elliptical path in the annular region between two circles, one inside other, with their centres separated by a certain distance:

Consider a smaller circle, with a radius r, centred at the origin (point O) completely inside the bigger one, with a radius R, centred at the point O' such that distance between their centres O & O' is d. Now consider an **elliptical path** (dotted curve A'BAB'A' with the centre C) traced by the centre of a third (parametric) circle, in the annular region between them, which touches the smaller circle externally & the bigger one internally.

For convenience, let's assume

$major\ axis,\ AA' = 2a\ \ \&\ \ minor\ axis,\ BB' = 2b$

Now, we have

$OE = OF = r,\ O'D = O'G = R\ \ \&\ \ OO' = d$
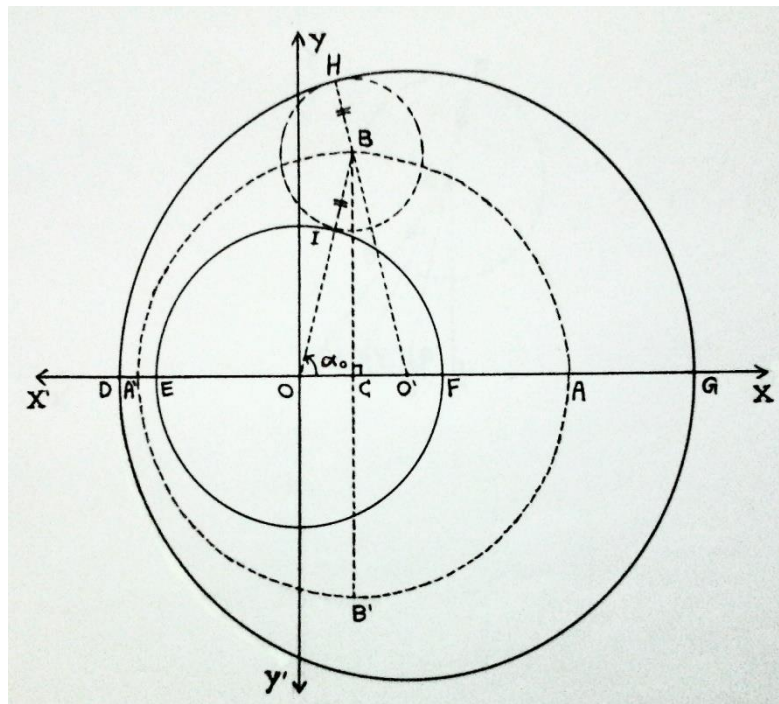
$$DE = O'D - O'E = O'D - (O'O + OE)$$
$$= R - (d + r)$$

Figure 1: Elliptical path (dotted curve A'BAB'A' with the centre C) is the locus of the centre of a third tangent (parametric) circle, in the annular region between two circles (smaller inside bigger) with their centres separated by a certain distance d, touching the smaller circle (with the centre O) externally & the bigger one (with the centre O') externally.

$$\therefore A'E = \frac{DE}{2} = \frac{R - (d + r)}{2} = \frac{R - d - r}{2}$$

$$FG = OG - OF = OO' + O'G - OF = d + R - r$$

$$\therefore FA = \frac{FG}{2} = \frac{R + d - r}{2}$$

$$\Rightarrow A'A = A'E + OE + OF + FA = \frac{R - d - r}{2} + r + r + \frac{R + d - r}{2} = \frac{R - d - r + 4r + R + d - r}{2} = R + r$$

$$\therefore Major\ axis, \qquad 2a = AA' = R + r \qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots..(I)$$

Now, join the vertex B of the transverse axis BCB' to the origin O so as the line OB intersects the smaller circle at the point I & then join B to the centre O' & extend the line O'B to intersect the bigger circle at the point H. Now, draw a third tangent circle, with a radius a & centre at the vertex B, which touches the smaller circle externally & the bigger one internally. Thus we have

$$BC = semi\ minor\ axis = b, \quad BI = BH = radius\ of\ tangent\ circle = a \quad \&$$

$$\Rightarrow OC = OG - CG = (OF + FG) - (CA + AG)$$

$$= (OF + FG) - \left(\frac{AA'}{2} + FA\right) \qquad \left(CA = CA' = \frac{AA'}{2} \ \& \ AG = FA\right)$$

$$OC = (r + R + d - r) - \left(\frac{R+r}{2} + \frac{R+d-r}{2}\right) = \frac{2R + 2d - 2R - d}{2} = \frac{d}{2}$$

$$\therefore \ CO' = OO' - OC = d - \frac{d}{2} = \frac{d}{2} \qquad \Rightarrow \ \boldsymbol{CO = CO' = \frac{d}{2}} \qquad \dots\dots\dots\dots\dots\dots (II)$$

Thus, the centre C is the midpoint of line segment OO'. Now in the right triangles BCO & BCO' we have

$$CO = CO' = \frac{d}{2}, \ \ BO = BO = b \ \& \ \angle BCO = \angle BCO' = 90^o$$

Hence, the right triangles BCO & BCO' are congruent.

In right ΔBCO

$$OB = \sqrt{(BC)^2 + (OC)^2} = \sqrt{b^2 + \left(\frac{d}{2}\right)^2} = \sqrt{\frac{4b^2 + d^2}{4}} = \frac{\sqrt{4b^2 + d^2}}{2}$$

$$\Rightarrow OB = OI + IB = r + a \ \therefore \ \boldsymbol{r + a = \frac{\sqrt{4b^2 + d^2}}{2}} \qquad \dots\dots\dots\dots\dots\dots. (III)$$

$$but, \ OB = O'B = O'H - BH = R - a \qquad (since, \ right\ triangles\ BCO \ \& \ BCO'\ are\ congruent)$$

Now, equating the value of OB from the eq(III), we get

$$R - a = r + a \ \Rightarrow 2a = R - r \ \ or \ \ a = \frac{R - r}{2}$$

Now, substituting the value of a in the above eq(III), we get

$$r + \frac{R - r}{2} = \frac{\sqrt{4b^2 + d^2}}{2} \ \ or \ \ \frac{\sqrt{4b^2 + d^2}}{2} = \frac{R + r}{2}$$

$$\Rightarrow \ 4b^2 + d^2 = (R + r)^2 \ \ or \ \ 2b = \sqrt{(R+r)^2 - d^2}$$

$$\therefore \ \boldsymbol{Minor\ axis}, \qquad \boldsymbol{2b = BB' = \sqrt{(R+r)^2 - d^2}} \qquad \dots\dots\dots\dots\dots\dots\dots. (IV)$$

$$\Rightarrow Eccentricity, \ e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \left(\frac{2b}{2a}\right)^2} = \sqrt{1 - \left(\frac{\sqrt{(R+r)^2 - d^2}}{(R+r)}\right)^2}$$

Mr H.C. Rajpoot (B. Tech, Mechanical Engineering from M.M.M. University of Technology, Gorakhpur-273010 (UP) India)

$$= \sqrt{1 - \frac{(R+r)^2 - d^2}{(R+r)^2}} = \sqrt{1 - 1 + \frac{d^2}{(R+r)^2}} = \sqrt{\frac{d^2}{(R+r)^2}} = \frac{d}{R+r}$$

$$\therefore \; \textbf{\textit{Eccentricity}}, \; e = \frac{d}{R+r} \quad (\, 0 \le e < 1 \quad \forall \; R \ge d + r) \qquad \ldots \ldots \ldots \ldots \ldots \ldots \; (V)$$

Equation of elliptical path: Since, the elliptical path has its centre at the point $C\left(\frac{d}{2}, 0\right)$, major axis $2a$ & minor axis $2b$, hence the equation of the elliptical path, assuming centre of the smaller circle as the origin, can be obtained by using **standard formula of ellipse centred at the origin** as follows

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1 \;\; \Rightarrow \;\; \frac{\left(x - \frac{d}{2}\right)^2}{a^2} + \frac{(y-0)^2}{b^2} = 1 \;\; or \;\; \frac{\left(x - \frac{d}{2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \textbf{\textit{Equation of elliptical path}}, \;\; \frac{\left(x - \frac{d}{2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \forall \; a = \frac{R+r}{2} \; \& \; b = \sqrt{(R+r)^2 - d^2}$$

Above equation is very useful for determining the co-ordinates of any point on the elliptical path & the radius of the parametric circle with centre at any arbitrary point on the elliptical path.

Now, the distance of foci from the centre C of elliptical path, can be determined as follows

$$X = \pm ae \;\; \Rightarrow \;\; x - \frac{d}{2} = \pm \left(\frac{R+r}{2}\right)\left(\frac{d}{R+r}\right) \;\; or \;\; x = \frac{d}{2} \pm \frac{d}{2} \;\; \Rightarrow \;\; x = 0 \; \& \; x = d$$

$$but \;\; x = 0 \; \& \; x = d \;\;\; represents \; the \; coordinates \; of \; the \; centres \; O \; \& \; O' \;\; respectively$$

Hence, it's obvious from the above values that **the centres of two circles are always the foci of the elliptical path in the annulus of these circles, smaller inside bigger, with their centres separated by a certain distance.**

3. Radius of the tangent circle with centre at the vertex B: The radius (a) of tangent circle, having centre at the vertex B of the transverse axis BCB' of the elliptical path, touching the smaller circle externally & the bigger one internally (See the above figure 1), is given as

$$a = \frac{R-r}{2} \qquad \forall \; R \ge d + r$$

And **angle α_o** of the line BO joining the centre B of the tangent circle to the origin O with the x-axis (line OO')

In right ΔBCO

$$tan\angle BOC = \frac{BC}{OC} \;\; \Rightarrow \;\; tan\alpha_o = \frac{b}{\left(\frac{d}{2}\right)} = \frac{2\left(\frac{\sqrt{(R+r)^2 - d^2}}{2}\right)}{d} = \frac{\sqrt{(R+r)^2 - d^2}}{d} = \sqrt{\left(\frac{R+r}{d}\right)^2 - 1}$$

$$or \;\; tan\alpha_o = \sqrt{\left(\frac{1}{e}\right)^2 - 1} = \sqrt{\frac{1 - e^2}{e^2}} = \frac{\sqrt{1 - e^2}}{e} \;\; or \;\; \alpha_o = \tan^{-1}\left(\frac{\sqrt{1 - e^2}}{e}\right)$$

$$\therefore \;\; \alpha_o = \tan^{-1}\left(\sqrt{\left(\frac{R+r}{d}\right)^2 - 1}\right) = \tan^{-1}\left(\frac{\sqrt{1 - e^2}}{e}\right) \quad \forall \; e = \frac{d}{R+r} \; \& \; R \ge d + r$$

Above, equation can be used to calculate angle α_o for the tangent circle with centre at the vertex B.

4. Derivation of the radius of tangent (parametric) circle touching the smaller circle externally &

the bigger one internally: Consider a **tangent (parametric) circle** with the centre at **any arbitrary point say E on the elliptical path** such that it touches the smaller circle at the point D externally & the bigger one at the point F internally. Now join the centre E of the tangent circle to the origin O, centre C & the centre O' by the dotted lines & also draw a perpendicular EM on the x-axis (see the figure 2) such that

$$\angle EOM = \alpha, \quad \angle ECM = \gamma \ \& \ \angle EO'M = \beta$$

Now, the **equation of line OE** passing through the origin O & inclined at an angle α with the positive direction of x-axis is given as follows

$$y = mx = x\tan\alpha \quad \forall \ \ 0 \le \alpha \le \pi$$

Now, solving the equation of the line OE & the equation of elliptical path to calculate the co-ordinates of the point of intersection E of the ellipse & the line by substituting $x = y\cot\alpha$ in the equation of elliptical path as follows
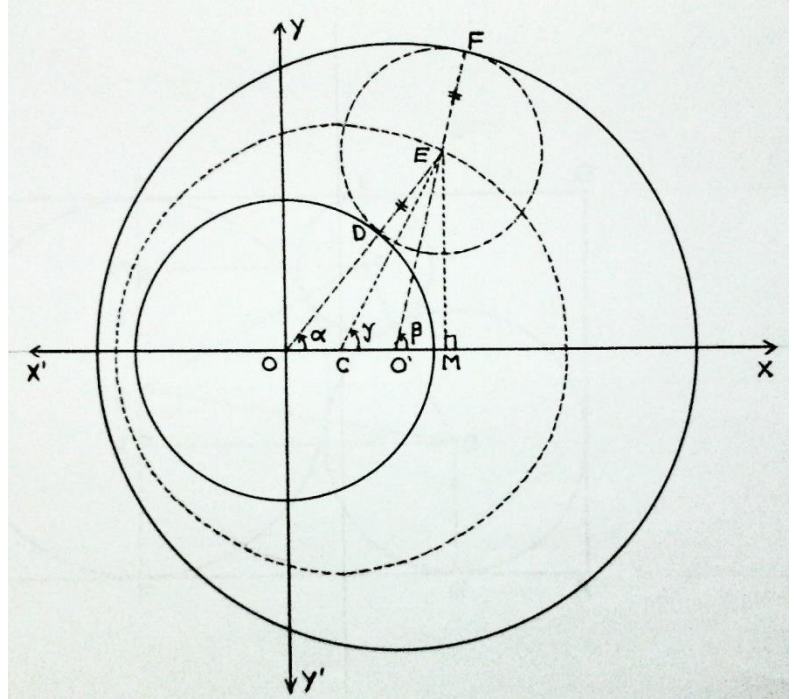

Figure 2: A tangent (parametric) circle, with the centre at any arbitrary point E on the elliptical path, is considered such that the angle of the line OE with the positive direction of x-axis is α in order to calculate its radius

$$\frac{\left(x - \frac{d}{2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1 \ \Rightarrow \ \frac{\left(y\cot\alpha - \frac{d}{2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{\left(y\cot\alpha - \frac{d}{2}\right)^2}{\left(\frac{R+r}{2}\right)^2} + \frac{y^2}{\left(\frac{\sqrt{(R+r)^2 - d^2}}{2}\right)^2} = 1 \quad (on \ substituting \ the \ values \ of \ a \ \& \ b)$$

$$or \ \left(\frac{\sqrt{(R+r)^2 - d^2}}{2}\right)^2 \left(y\cot\alpha - \frac{d}{2}\right)^2 + \left(\frac{R+r}{2}\right)^2 y^2 = \left(\frac{R+r}{2}\right)^2 \left(\frac{\sqrt{(R+r)^2 - d^2}}{2}\right)^2$$

$$\Rightarrow \left(\frac{(R+r)^2 - d^2}{4}\right)\left(y^2\cot^2\alpha + \frac{d^2}{4} - ydcot\alpha\right) + \frac{(R+r)^2}{4}y^2 = \frac{(R+r)^2}{4}\left(\frac{(R+r)^2 - d^2}{4}\right)$$

$$\Rightarrow \left\{\left(\frac{(R+r)^2 - d^2}{4}\right)\cot^2\alpha + \frac{(R+r)^2}{4}\right\}y^2 - \left\{\left(\frac{(R+r)^2 - d^2}{4}\right)dcot\alpha\right\}y + \left(\frac{(R+r)^2 - d^2}{4}\right)\frac{d^2}{4}$$
$$- \frac{(R+r)^2}{4}\left(\frac{(R+r)^2 - d^2}{4}\right) = 0$$

$$\left\{\frac{(R+r)^2\cosec^2\alpha}{4} - \frac{d^2}{4}\cot^2\alpha\right\}y^2 - \left\{\left(\frac{(R+r)^2 - d^2}{4}\right)dcot\alpha\right\}y - \left(\frac{(R+r)^2 - d^2}{4}\right)\left(\frac{(R+r)^2}{4} - \frac{d^2}{4}\right) = 0$$

$$\Rightarrow \left\{\frac{(R+r)^2\cosec^2\alpha}{4} - \frac{d^2}{4}\cot^2\alpha\right\}y^2 - \left\{\left(\frac{(R+r)^2 - d^2}{4}\right)dcot\alpha\right\}y - \left(\frac{(R+r)^2 - d^2}{4}\right)^2 = 0$$

Now, solving above quadratic for the values of y as follows

y

$$= \frac{\left(\frac{(R+r)^2 - d^2}{4}\right)dcot\alpha \pm \sqrt{\left\{\left(\frac{(R+r)^2 - d^2}{4}\right)dcot\alpha\right\}^2 + 4\left\{\frac{(R+r)^2cosec^2\alpha}{4} - \frac{d^2}{4}cot^2\alpha\right\}\left(\frac{(R+r)^2 - d^2}{4}\right)^2}}{2\left\{\frac{(R+r)^2cosec^2\alpha}{4} - \frac{d^2}{4}cot^2\alpha\right\}}$$

$$= \frac{\left(\frac{(R+r)^2 - d^2}{4}\right)dcot\alpha \pm \left(\frac{(R+r)^2 - d^2}{4}\right)\sqrt{d^2cot^2\alpha + (R+r)^2cosec^2\alpha - d^2cot^2\alpha}}{\frac{1}{2}\{(R+r)^2cosec^2\alpha - d^2cot^2\alpha\}}$$

$$= 2\left(\frac{(R+r)^2 - d^2}{4}\right)\left(\frac{dcot\alpha \pm (R+r)cosec\alpha}{(R+r)^2cosec^2\alpha - d^2cot^2\alpha}\right) = \left(\frac{(R+r)^2 - d^2}{2}\right)\left(\frac{dcos\alpha \pm (R+r)}{(R+r)^2 - d^2cos^2\alpha}\right)sin\alpha$$

Taking positive sign, we get

$$y = \left(\frac{(R+r)^2 - d^2}{2}\right)\left(\frac{dcos\alpha + (R+r)}{(R+r)^2 - d^2cos^2\alpha}\right)sin\alpha$$

$$= \left(\frac{(R+r)^2 - d^2}{2}\right)\left(\frac{dcos\alpha + (R+r)}{((R+r) + dcos\alpha)((R+r) - dcos\alpha)}\right)sin\alpha = \frac{sin\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right)$$

$$\Rightarrow y = \frac{sin\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right) \geq 0 \quad \forall \ 0 \leq \alpha \leq \pi \ \textit{hence this value of y is accepted}$$

$$\therefore x = ycot\alpha = \frac{sin\alpha cot\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right) = \frac{cos\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right)$$

Taking negative sign, we get

$$y = \left(\frac{(R+r)^2 - d^2}{2}\right)\left(\frac{dcos\alpha - (R+r)}{(R+r)^2 - d^2cos^2\alpha}\right)sin\alpha$$

$$= \left(\frac{(R+r)^2 - d^2}{2}\right)\left(\frac{dcos\alpha - (R+r)}{((R+r) + dcos\alpha)((R+r) - dcos\alpha)}\right)sin\alpha = \frac{-sin\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) + dcos\alpha}\right)$$

$$\Rightarrow y = \frac{-sin\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) + dcos\alpha}\right) \leq 0 \quad \forall \ 0 \leq \alpha \leq \pi \ \textit{hence this value of y is discarded}$$

Hence, the co-ordinates of the point of intersection E (i.e. the centre of tangent circle) is given as

$$\boxed{E \equiv (x, y) \equiv \left(\frac{cos\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right), \frac{sin\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right)\right) \quad \forall \ 0 \leq \alpha \leq \pi}$$

Now, in the right ΔEMO, we have

$$OM = x = \frac{cos\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right) \ \& \ EM = y = \frac{sin\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right)$$

$$\therefore OE = \sqrt{(OM)^2 + (EM)^2} = \sqrt{\left(\frac{cos\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right)\right)^2 + \left(\frac{sin\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - dcos\alpha}\right)\right)^2}$$

$$\Rightarrow OE = \frac{1}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - d\cos\alpha}\right)\sqrt{\sin^2\alpha + \cos^2\alpha} = \frac{1}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - d\cos\alpha}\right)$$

$$but, OE = OD + DE = r + ED \quad \Rightarrow \quad ED = OE - r$$

$$\therefore ED = \frac{1}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - d\cos\alpha}\right) - r = \frac{1}{2}\left(\frac{(R+r)^2 - d^2 - 2r(R+r) + 2rd\cos\alpha}{(R+r) - d\cos\alpha}\right)$$

$$= \frac{1}{2}\left(\frac{R^2 + r^2 + 2rR - d^2 - 2rR - 2r^2 + 2rd\cos\alpha}{(R+r) - d\cos\alpha}\right) = \frac{1}{2}\left(\frac{R^2 - d^2 - r^2 + 2rd\cos\alpha}{(R+r) - d\cos\alpha}\right)$$

Hence, the **radius (R_α) of the tangent (parametric) circle** for the given value of the angle α is given as follows

$$R_\alpha = \frac{1}{2}\left(\frac{R^2 - d^2 - r^2 + 2rd\cos\alpha}{(R+r) - d\cos\alpha}\right) \quad \forall \ 0 \le \alpha \le \pi \ \& \ R \ge d + r$$

The above expression is very useful to determine the radius (R_α) of the tangent (parametric) circle for the given value of parametric angle α.

5. Parametric angles β & **γ :** The parametric angles β & γ are determined for the given value of parametric angle α as follows

In the right ΔEMC, we have

$$tan\angle ECM = \frac{EM}{CM} = \frac{EM}{OM - OC} \quad \Rightarrow \quad tan\gamma = \frac{\frac{\sin\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - d\cos\alpha}\right)}{\frac{\cos\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - d\cos\alpha}\right) - \frac{d}{2}}$$

$$= \frac{((R+r)^2 - d^2)\sin\alpha}{(R+r)^2\cos\alpha - d^2\cos\alpha - (R+r)d + d^2\cos\alpha} = \frac{((R+r)^2 - d^2)\sin\alpha}{(R+r)^2\cos\alpha - (R+r)d}$$

$$\therefore \ tan\gamma = \frac{((R+r)^2 - d^2)\sin\alpha}{(R+r)^2\cos\alpha - (R+r)d} \quad 0 \le \gamma \le \pi \ \forall \ 0 \le \alpha \le \pi$$

In the right $\Delta EMO'$, we have

$$tan\angle EO'M = \frac{EM}{O'M} = \frac{EM}{OM - OO'} \quad \Rightarrow \quad tan\beta = \frac{\frac{\sin\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - d\cos\alpha}\right)}{\frac{\cos\alpha}{2}\left(\frac{(R+r)^2 - d^2}{(R+r) - d\cos\alpha}\right) - d}$$

$$= \frac{((R+r)^2 - d^2)\sin\alpha}{(R+r)^2\cos\alpha - d^2\cos\alpha - 2(R+r)d + 2d^2\cos\alpha} = \frac{((R+r)^2 - d^2)\sin\alpha}{(R+r)^2\cos\alpha + d^2\cos\alpha - 2(R+r)d}$$

$$\therefore \ tan\beta = \frac{((R+r)^2 - d^2)\sin\alpha}{((R+r)^2 + d^2)\cos\alpha - 2(R+r)d} \quad 0 \le \beta \le \pi \ \forall \ 0 \le \alpha \le \pi$$

Thus, the parametric angles β & γ can be determined for the given value of angle α. Also these angles can be co-related with each other by using above expressions.

Conclusion: For given two circles, smaller inside the bigger one, with their radii r & R respectively and their centres separated by a certain distance d ($\forall$ $d \geq 0$), the **locus of the centre of a third tangent (parametric) circle** (in the annulus of two circles) touching the smaller circle externally & the bigger one internally, is always an **ellipse** (except circle in case of concentric circles) then all the parameters of this elliptical path are determined as follows

Major axis $(2a)$:

$$2a = R + r$$

Minor axis $(2b)$:

$$2b = \sqrt{(R+r)^2 - d^2}$$

Eccentricity (e):

$$e = \frac{d}{R+r} \quad (\ 0 \leq e < 1 \quad \forall\ R \geq d + r)$$

Radius (R_α) **of the tangent (parametric) circle:** For the given value of the parametric angle α

$$R_\alpha = \frac{1}{2}\left(\frac{R^2 - d^2 - r^2 + 2rd\cos\alpha}{(R+r) - d\cos\alpha}\right) \quad \forall\ 0 \leq \alpha \leq \pi\ \&\ R \geq d + r$$

Parametric angles β & γ :

$$tan\beta = \frac{((R+r)^2 - d^2)\sin\alpha}{((R+r)^2 + d^2)\cos\alpha - 2(R+r)d} \quad 0 \leq \beta \leq \pi \quad \forall\ 0 \leq \alpha \leq \pi$$

$$tan\gamma = \frac{((R+r)^2 - d^2)\sin\alpha}{(R+r)^2\cos\alpha - (R+r)d} \quad 0 \leq \gamma \leq \pi \quad \forall\ 0 \leq \alpha \leq \pi$$

*All the articles above have been derived by the author by using **simple geometry & trigonometry**. All above articles (formula) are very practical & simple to apply in case studies & practical applications of 2-D Geometry. These articles are the most generalised expressions which can be used for the analysis of the elliptical path in the annular region between two circles, smaller inside the bigger one & their centres separated by a certain distance in order to calculate major axis, minor axis, eccentricity & the radius of the tangent (parametric) circle.*

Note: Above articles had been derived & *illustrated by **Mr H.C. Rajpoot (B Tech, Mechanical Engineering)***

M.M.M. University of Technology, Gorakhpur-273010 (UP) India   *Feb, 2015*
Email:rajpootharishchandra@gmail.com

Author's Home Page: https://notionpress.com/author/HarishChandraRajpoot