$Mathematical\ Analysis\ of\ Decahedron\ With\ Right\ Kite\ Faces$

$Application\ of\ HCR's\ Theory\ of\ Polygon$

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Introduction: We are here to analyse a decahedron having 10 congruent faces each as a **right kite** (**i.e. cyclic quadrilateral consisting of two congruent right triangles with a hypotenuse in common**) & 12 vertices lying on a spherical surface with a certain radius. All 10 right kite faces are at an equal distance from the centre of the decahedron. Each of 10 right kite faces always has two right angles, one acute angle (α) $\left(\forall\ \boldsymbol{\alpha = 2\tan^{-1}\left(\sqrt{\sqrt{5}-2}\right) \approx\ 51.83^o}\right)$ & one obtuse angle (β) $(\forall\ \boldsymbol{\beta = 180^o - \alpha \approx 128.17^o}).$ Its each face has two pairs of unequal sides $\boldsymbol{a}$ & $\boldsymbol{b}$ $\left(\forall\ \boldsymbol{a = b\sqrt{\sqrt{5}-2}}\right)$ & can be divided into two congruent right triangles having longer diagonal of the face as their common hypotenuse. It has two identical & diagonally opposite vertices (say vertices C & E out of total 12 vertices) at each of which five right kite faces meet together & rest 10 vertices are identical at each of which three right kite faces meet together (See the figure 1).

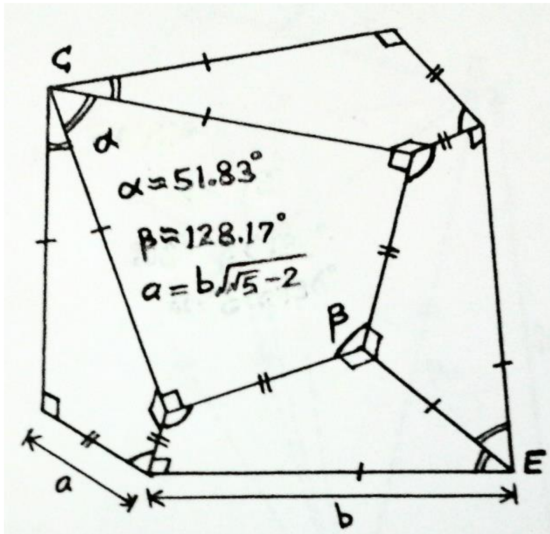

Figure 1: A decahedron having 12 vertices & 10 congruent faces each as a right kite having two pairs of unequal sides $\boldsymbol{a}$ & $\boldsymbol{b}$ $\left(\forall\ \boldsymbol{a = b\sqrt{\sqrt{5}-2}}\right)$, two right angles, one acute angle $\boldsymbol{\alpha}$ $(\forall\ \boldsymbol{\alpha \approx 51.83^o})$ & one obtuse angle $\boldsymbol{\beta}$ $(\forall\ \boldsymbol{\beta = 180^o - \alpha \approx 128.17^o})$

Analysis of decahedron: Let there be a **decahedron** having 10 congruent faces each as a **right kite having two pairs of unequal sides** a & b $(\forall\ a < b)$. Now let's first determine the relation between unequal sides of the right kite face by calculating the ratio of unequal sides a & b.

Derivation of relation between unequal sides $\boldsymbol{a}$ & $\boldsymbol{b}$ of each right kite face of the decahedron: Let h be the normal distance of each of 10 congruent right kite faces of a decahedron. Now draw a perpendicular OO' from the centre O of the decahedron at the point O' to any of its right kite faces say face ABCD. Since the right kite face ABCD is a cyclic quadrilateral hence all its vertices A, B, C & D lie on the circle & the perpendicular OO' will have its foot O' at the centre of the circumscribed circle. (See figure 2 below). Now join all the vertices A, B, C & D to the centre O' to obtain two congruent right triangles ΔABC & $\Delta\ ADC$. Thus we have,

$$AB = AD = a\ \ \&\ \ BC = CD = b \quad \forall\ a < b$$

$$\Rightarrow\ O'A = O'B = O'C = O'D = \frac{AC}{2} = \frac{\sqrt{(AB)^2 + (BC)^2}}{2} = \frac{\sqrt{a^2 + b^2}}{2}$$

Now, draw the perpendiculars O'M & O'N to the sides AB & BC at their mid-points M & N respectively. Thus isosceles $\Delta AO'B$ is divided into two congruent right triangles $\Delta O'MA$ & $\Delta O'MB$. Similarly, isosceles $\Delta BO'C$ is divided into two congruent right triangles $\Delta O'NB$ & $\Delta O'NC$.

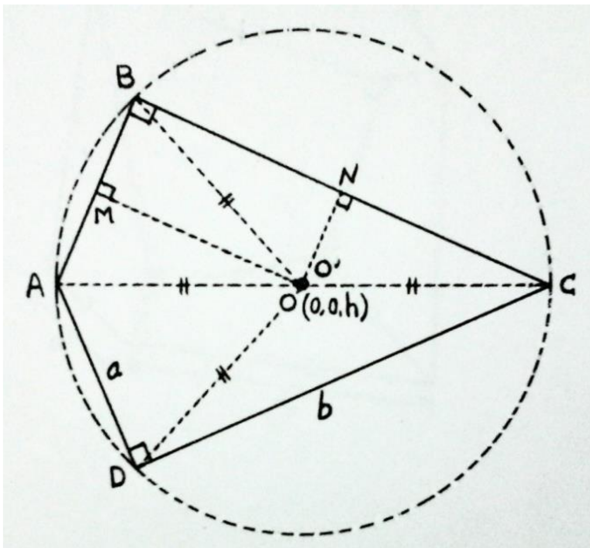

Figure 2: A perpendicular OO' (normal to the plane of paper) is drawn from the centre $\boldsymbol{O\ (0,0,h)}$ of the decahedron to the right kite face ABCD at the circumscribed centre O' of face ABCD with $\boldsymbol{AB = AD = a}$ & $\boldsymbol{BC = CD = b}\ \ \forall\ \boldsymbol{a < b\tan 36^o}$

In right $\Delta O'MA$

$$O'M = \sqrt{(O'A)^2 - (AM)^2} = \sqrt{\left(\frac{\sqrt{a^2+b^2}}{2}\right)^2 - \left(\frac{a}{2}\right)^2} = \frac{b}{2}$$

Similarly, in right $\Delta O'NB$

$$O'N = \sqrt{(O'B)^2 - (BN)^2} = \sqrt{\left(\frac{\sqrt{a^2+b^2}}{2}\right)^2 - \left(\frac{b}{2}\right)^2} = \frac{a}{2}$$

We know from **HCR's Theory of Polygon** that the **solid angle ($\boldsymbol{\omega}$), subtended by a right triangle having orthogonal sides $\boldsymbol{a}$ & $\boldsymbol{b}$ at any point at a normal distance $\boldsymbol{h}$ on the vertical axis passing through the common vertex of the side b & hypotenuse,** is given by **HCR's Standard Formula-1** as follows

$$\boldsymbol{\omega = \sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left\{\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left(\frac{h}{\sqrt{h^2+b^2}}\right)\right\}}$$

Hence, **solid angle ($\boldsymbol{\omega_{\Delta O'MA}}$)** subtended by **right $\boldsymbol{\Delta O'MA}$** at the centre O $(0, 0, h)$ of decahedron is given as

$$\omega_{\Delta O'MA} = \sin^{-1}\left(\frac{(AM)}{\sqrt{(AM)^2 + (O'M)^2}}\right) - \sin^{-1}\left\{\left(\frac{(AM)}{\sqrt{(AM)^2 + (O'M)^2}}\right)\left(\frac{(OO')}{\sqrt{(OO')^2 + (O'M)^2}}\right)\right\}$$

Now, by substituting all the corresponding values in the above expression we have

$$\omega_{\Delta O'MA} = \sin^{-1}\left(\frac{\left(\frac{a}{2}\right)}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}}\right) - \sin^{-1}\left\{\left(\frac{\left(\frac{a}{2}\right)}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}}\right)\left(\frac{h}{\sqrt{h^2 + \left(\frac{b}{2}\right)^2}}\right)\right\}$$

$$= \sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left\{\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left(\frac{2h}{\sqrt{4h^2+b^2}}\right)\right\}$$

$$\therefore\ \boldsymbol{\omega_{\Delta O'MA} = \sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left(\frac{2ah}{\sqrt{(a^2+b^2)(4h^2+b^2)}}\right)} \quad \dots\dots\dots\dots\dots(I)$$

Similarly, **solid angle ($\boldsymbol{\omega_{\Delta O'NB}}$)** subtended by **right $\boldsymbol{\Delta O'NB}$** at the centre O $(0, 0, h)$ of decahedron is given as

$$\omega_{\Delta O'NB} = \sin^{-1}\left(\frac{(BN)}{\sqrt{(BN)^2 + (O'N)^2}}\right) - \sin^{-1}\left\{\left(\frac{(BN)}{\sqrt{(BN)^2 + (O'N)^2}}\right)\left(\frac{(OO')}{\sqrt{(OO')^2 + (O'N)^2}}\right)\right\}$$

$$\omega_{\Delta O'NB} = \sin^{-1}\left(\frac{\left(\frac{b}{2}\right)}{\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{2}\right)^2}}\right) - \sin^{-1}\left\{\left(\frac{\left(\frac{b}{2}\right)}{\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{2}\right)^2}}\right)\left(\frac{h}{\sqrt{h^2 + \left(\frac{a}{2}\right)^2}}\right)\right\}$$

$$= \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{a^2+b^2}}\right)\left(\frac{2h}{\sqrt{4h^2+a^2}}\right)\right\}$$

$$\therefore\ \boldsymbol{\omega_{\Delta O'NB} = \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left(\frac{2bh}{\sqrt{(a^2+b^2)(4h^2+a^2)}}\right)} \quad \dots\dots\dots\dots\dots(II)$$

Now, solid angle (ω_{ABCD}) subtended by the right kite face ABCD at the centre O $(0, 0, h)$ of the decahedron is given as

$$\boldsymbol{\omega_{ABCD}} = \omega_{\Delta ABC} + \omega_{\Delta ADC} = 2(\omega_{\Delta ABC}) \quad (since,\ \ \Delta ABC\ \&\Delta ADC\ are\ congruents\ )$$

$$= 2(\omega_{\Delta AO'B} + \omega_{\Delta BO'C}) = 2\{(\omega_{\Delta O'MA} + \omega_{\Delta O'MB}) + (\omega_{\Delta O'NB} + \omega_{\Delta O'NC})\}$$

$$= 2\{2(\omega_{\Delta O'MA}) + 2(\omega_{\Delta O'NB})\} = \mathbf{4(\boldsymbol{\omega}_{\Delta O'MA} + \boldsymbol{\omega}_{\Delta O'NB})} \quad (by\ congruent\ triangles)$$

Since all 10 right kite faces of decahedron are congruent hence the **solid angle subtended by each face at the centre of decahedron** is

$$= \frac{total\ solid\ angle}{no.of\ congruent\ right\ kite\ faces} = \frac{4\pi}{10} = \frac{2\pi}{5}$$

$$\Rightarrow\ \boldsymbol{\omega_{ABCD}} = \frac{\mathbf{2\pi}}{\mathbf{5}} \quad or \quad 4(\omega_{\Delta O'MA} + \omega_{\Delta O'NB}) = \frac{2\pi}{5} \quad or \quad \omega_{\Delta O'MA} + \omega_{\Delta O'NB} = \frac{2\pi}{20} = \frac{\pi}{10}$$

$$\therefore\ \boldsymbol{\omega_{\Delta O'MA} + \omega_{\Delta O'NB}} = \frac{\boldsymbol{\pi}}{\mathbf{10}}$$

Now, by setting the corresponding values from the eq(I) & (II) in the above expression, we get

$$\sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left(\frac{2ah}{\sqrt{(a^2+b^2)(4h^2+b^2)}}\right) + \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left(\frac{2bh}{\sqrt{(a^2+b^2)(4h^2+a^2)}}\right) = \frac{\pi}{10}$$

$$\Rightarrow \left[\sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) + \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)\right] - \left[\sin^{-1}\left(\frac{2ah}{\sqrt{(a^2+b^2)(4h^2+b^2)}}\right) + \sin^{-1}\left(\frac{2bh}{\sqrt{(a^2+b^2)(4h^2+a^2)}}\right)\right] = \frac{\pi}{10}$$

$$\Rightarrow \left[\sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\sqrt{1-\left(\frac{b}{\sqrt{a^2+b^2}}\right)^2} + \frac{b}{\sqrt{a^2+b^2}}\sqrt{1-\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2}\right)\right] - \left[\sin^{-1}\left(\frac{2ah}{\sqrt{(a^2+b^2)(4h^2+b^2)}}\sqrt{1-\left(\frac{2bh}{\sqrt{(a^2+b^2)(4h^2+a^2)}}\right)^2} + \frac{2bh}{\sqrt{(a^2+b^2)(4h^2+a^2)}}\sqrt{1-\left(\frac{2ah}{\sqrt{(a^2+b^2)(4h^2+b^2)}}\right)^2}\right)\right] = \frac{\pi}{10}$$

$$\Rightarrow\ \sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}} \times \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}} \times \frac{b}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left(\frac{2ah}{\sqrt{(a^2+b^2)(4h^2+b^2)}}\sqrt{\frac{4a^2h^2+4b^2h^2+a^4+a^2b^2-4b^2h^2}{(a^2+b^2)(4h^2+a^2)}} + \frac{2bh}{\sqrt{(a^2+b^2)(4h^2+a^2)}}\sqrt{\frac{4a^2h^2+4b^2h^2+a^2b^2+b^4-4a^2h^2}{(a^2+b^2)(4h^2+b^2)}}\right) = \frac{\pi}{10}$$

$$\Rightarrow\ \sin^{-1}\left(\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}\right) - \sin^{-1}\left(\frac{2ah\sqrt{a^2(4h^2+a^2+b^2)}}{(a^2+b^2)\sqrt{(4h^2+a^2)(4h^2+b^2)}} + \frac{2bh\sqrt{b^2(4h^2+a^2+b^2)}}{(a^2+b^2)\sqrt{(4h^2+a^2)(4h^2+b^2)}}\right) = \frac{\pi}{10}$$

$$\Rightarrow\ \sin^{-1}\left(\frac{a^2+b^2}{a^2+b^2}\right) - \sin^{-1}\left(\frac{2h(a^2+b^2)\sqrt{(4h^2+a^2+b^2)}}{(a^2+b^2)\sqrt{(4h^2+a^2)(4h^2+b^2)}}\right) = \frac{\pi}{10}$$

$$\Rightarrow \sin^{-1}(1) - \sin^{-1}\left(2h\sqrt{\frac{4h^2+a^2+b^2}{(4h^2+a^2)(4h^2+b^2)}}\right) = \frac{\pi}{10}$$

$$\Rightarrow \sin^{-1}\left(2h\sqrt{\frac{4h^2+a^2+b^2}{(4h^2+a^2)(4h^2+b^2)}}\right) = \sin^{-1}(1) - \frac{\pi}{10} = \frac{\pi}{2} - \frac{\pi}{10}$$

$$\Rightarrow 2h\sqrt{\frac{4h^2+a^2+b^2}{(4h^2+a^2)(4h^2+b^2)}} = sin\left(\frac{\pi}{2} - \frac{\pi}{10}\right) = cos\frac{\pi}{10} = cos18^o$$

$$\Rightarrow \left(2h\sqrt{\frac{4h^2+a^2+b^2}{(4h^2+a^2)(4h^2+b^2)}}\right)^2 = (cos18^o)^2 = cos^2 18^o$$

$$\Rightarrow \frac{4h^2(4h^2+a^2+b^2)}{(4h^2+a^2)(4h^2+b^2)} = cos^2 18^o$$

$$\Rightarrow \ 16h^4 + 4(a^2+b^2)h^2 = (16h^4 + 4(a^2+b^2)h^2 + a^2b^2)cos^2 18^o$$

$$\Rightarrow \ 16(1 - cos^2 18^o)h^4 + 4(a^2+b^2)(1 - cos^2 18^o)h^2 - a^2b^2cos^2 18^o = 0$$

$$\Rightarrow \ 16sin^2 18^o h^4 + 4sin^2 18^o(a^2+b^2)h^2 - a^2b^2cos^2 18^o = 0$$

$$\Rightarrow \ \boldsymbol{16h^4 + 4(a^2+b^2)h^2 - a^2b^2cot^2 18^o = 0}$$

Now, solving the above bi-quadratic equation to obtain the values of h^2 as follows

$$\Rightarrow \ h^2 = \frac{-4(a^2+b^2) \pm \sqrt{\left(-4(a^2+b^2)\right)^2 + 4(16)a^2b^2cot^2 18^o}}{2(16)}$$

$$= \frac{-4(a^2+b^2) \pm 4\sqrt{(a^2+b^2)^2 + 4a^2b^2cot^2 18^o}}{32}$$

$$= \frac{-(a^2+b^2) \pm \sqrt{(a^2+b^2)^2 + 4\left(5+2\sqrt{5}\right)a^2b^2}}{8} \qquad \left(since, \ \boldsymbol{cot^2 18^o = 5 + 2\sqrt{5}}\right)$$

Since, $h > 0 \ \ or \ h^2 > 0$ hence, by taking positive sign we get the required value of normal height h as follows

$$h^2 = \frac{-(a^2+b^2) + \sqrt{(a^2+b^2)^2 + 4\left(5+2\sqrt{5}\right)a^2b^2}}{8} = \frac{\sqrt{\left\{(a^2+b^2)^2 + 4\left(5+2\sqrt{5}\right)a^2b^2\right\}} - (a^2+b^2)}{8}$$

$$\boldsymbol{Normal\ distance,\ \ h} = \frac{\sqrt{\sqrt{\left\{(\boldsymbol{a^2+b^2})^2 + \boldsymbol{4}\left(\boldsymbol{5+2\sqrt{5}}\right)\boldsymbol{a^2b^2}\right\}} - (\boldsymbol{a^2+b^2})}}{\boldsymbol{2\sqrt{2}}} \quad \ldots\ldots\ldots\ldots\ldots..(III)$$

Outer (circumscribed) radius (R_o) of decahedron (i.e. radius of the spherical surface passing through all 12 vertices of the decahedron): Let R_o be the circumscribed radius i.e. the radius of the spherical surface passing through all 12 vertices of a decahedron. (See figure 3)

Consider the right kite face ABCD & join all its vertices A, B, C & D to the centre O of the decahedron. Draw a perpendicular OO' from the centre O to the face ABCD at the point O'. Since the spherical surface with a radius R_o is passing though all 12 vertices of decahedron hence we have

$$OA = OB = OC = OD = R_o \ \ \& \ \ OO' = h$$

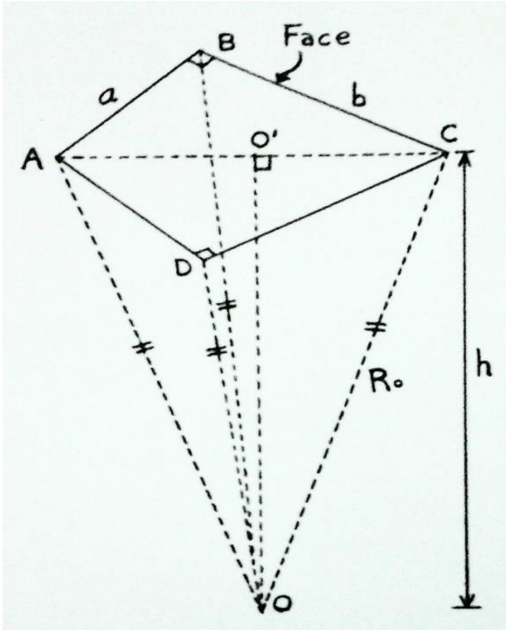

Figure 3: An elementary right pyramid OABCD is obtained by joining all four vertices A, B, C & D of right kite face ABCD to the centre O of a decahedron.

Now, in right $\Delta OO'A$

$$\Rightarrow (OA)^2 = (OO')^2 + (O'A)^2 = h^2 + (O'A)^2$$

$$= \left(\frac{\sqrt{\sqrt{\{(a^2+b^2)^2 + 4(5+2\sqrt{5})a^2b^2\}} - (a^2+b^2)}}{2\sqrt{2}}\right)^2 + \left(\frac{\sqrt{a^2+b^2}}{2}\right)^2$$

$$= \frac{\sqrt{\{(a^2+b^2)^2 + 4(5+2\sqrt{5})a^2b^2\}} - (a^2+b^2)}{8} + \frac{a^2+b^2}{4}$$

$$= \frac{\sqrt{\{(a^2+b^2)^2 + 4(5+2\sqrt{5})a^2b^2\}} - (a^2+b^2) + 2(a^2+b^2)}{8}$$

$$\Rightarrow {R_o}^2 = \frac{\sqrt{\{(a^2+b^2)^2 + 4(5+2\sqrt{5})a^2b^2\}} + a^2 + b^2}{8}$$

$$\Rightarrow \mathbf{8{R_o}^2} = \sqrt{\{(\boldsymbol{a^2+b^2})^2 + \mathbf{4}(\mathbf{5}+\mathbf{2}\sqrt{\mathbf{5}})\boldsymbol{a^2b^2}\}} + \boldsymbol{a^2} + \boldsymbol{b^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (IV)$$

Since, all 12 vertices are located on the circumscribed spherical surface, **Let us consider a great circles with the centre 'O' passing through two identical & diagonally opposite vertices C & E** (as shown in the figure 4 below). Hence the line CE is a diametric line passing through the centre O of a great circle (on the **circumscribed spherical surface**) & the vertices C, A & E are lying on the (great) circle hence, the $\angle CAE = 90^o$. Now

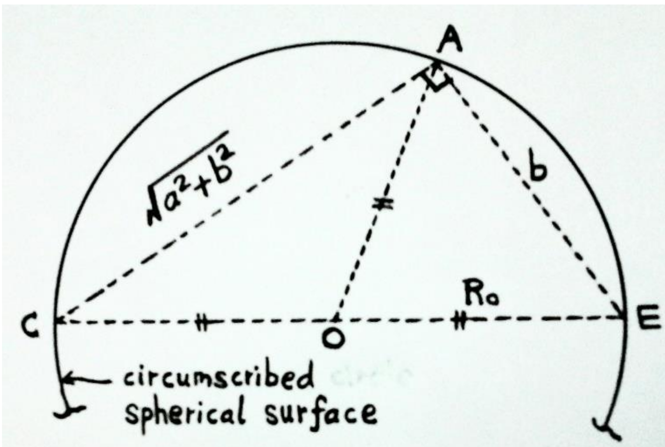

Figure 4: Two vertices C & E are identical & diagonally opposite. Vertices C, A & E are lying on a great circle on the circumscribed spherical surface & $\angle CAE = 90^o$

In right ΔCAE

$$(CE)^2 = (AC)^2 + (AE)^2$$

Now, by substituting all the corresponding values in the above expression we get

$$(2R_o)^2 = \left(\sqrt{a^2+b^2}\right)^2 + (b)^2$$

$$4{R_o}^2 = a^2 + b^2 + b^2 = a^2 + 2b^2 \;\Rightarrow\; \mathbf{8{R_o}^2 = 2a^2 + 4b^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(V)$$

Now, equating eq(IV) & (V), we get

$$2a^2 + 4b^2 = \sqrt{\{(a^2+b^2)^2 + 4(5+2\sqrt{5})a^2b^2\}} + a^2 + b^2$$

$$\Rightarrow\; a^2 + 3b^2 = \sqrt{\{(a^2+b^2)^2 + 4(5+2\sqrt{5})a^2b^2\}}$$

$$\Rightarrow\; (a^2+3b^2)^2 = (a^2+b^2)^2 + 4(5+2\sqrt{5})a^2b^2$$

$$\Rightarrow\; a^4 + 9b^4 + 6a^2b^2 = a^4 + b^4 + 2a^2b^2 + 4(5+2\sqrt{5})a^2b^2$$

$$\Rightarrow\; 8b^4 + 4a^2b^2 - 4(5+2\sqrt{5})a^2b^2 = 0 \;\; or \;\; 8b^4 - 4(5+2\sqrt{5}-1)a^2b^2 = 0$$

$$or\;\; 8b^4 - 8(2+\sqrt{5})a^2b^2 = 0 \;\Rightarrow\; 8b^2\left(b^2 - (2+\sqrt{5})a^2\right) = 0$$

But, $b \neq 0$ hence, we have

$$b^2 - (2+\sqrt{5})a^2 = 0 \;\; or \;\; \frac{a^2}{b^2} = \frac{1}{2+\sqrt{5}} \;\; or \;\; \frac{a}{b} = \sqrt{\frac{1}{2+\sqrt{5}}}$$

$$\frac{a}{b} = \sqrt{\frac{(\sqrt{5}-2)}{(\sqrt{5}+2)(\sqrt{5}-2)}} = \sqrt{\frac{(\sqrt{5}-2)}{(5-4)}} = \sqrt{\sqrt{5}-2}$$

$$\therefore \textit{ The required relation between unequal sides}, \boldsymbol{a \;\&\; b} \;\Rightarrow\; \boldsymbol{a = b\sqrt{\sqrt{5}-2} \approx 0.485868271b}$$

In right ΔABC (from the figure2)

$$tan\angle ACB = \frac{AB}{BC} = \frac{a}{b} \;\Rightarrow\; tan\frac{\alpha}{2} = \frac{a}{b} = \sqrt{\sqrt{5}-2} \;\; or \;\; \alpha = 2\tan^{-1}\sqrt{\sqrt{5}-2}$$

$$\therefore \;\; \boldsymbol{acute\ angle, \alpha = 2\tan^{-1}\sqrt{\sqrt{5}-2} \approx 51.83^o}$$

$$\therefore \;\; \boldsymbol{obtuse\ angle, \beta = 180^o - \alpha \approx 128.17^o}$$

Now, setting the value of a in term of b in the eq(III), we get

$$h = \frac{\sqrt{\sqrt{\{(a^2+b^2)^2 + 4(5+2\sqrt{5})a^2b^2\}} - (a^2+b^2)}}{2\sqrt{2}}$$

$$= \frac{\sqrt{\sqrt{\left\{\left(\left(b\sqrt{\sqrt{5}-2}\right)^2 + b^2\right)^2 + 4(5+2\sqrt{5})\left(b\sqrt{\sqrt{5}-2}\right)^2 b^2\right\}} - \left(\left(b\sqrt{\sqrt{5}-2}\right)^2 + b^2\right)}}{2\sqrt{2}}$$

$$=\frac{\sqrt{\sqrt{\left\{\left(b^2(\sqrt{5}-1)\right)^2+4(5+2\sqrt{5})(\sqrt{5}-2)b^4\right\}}-\left(b^2(\sqrt{5}-1)\right)}}{2\sqrt{2}}$$

$$=\frac{\sqrt{b^2\sqrt{\{5+1-2\sqrt{5}+4(\sqrt{5})\}}-b^2(\sqrt{5}-1)}}{2\sqrt{2}}=\frac{\sqrt{b^2\sqrt{\{5+1+2\sqrt{5}\}}-b^2(\sqrt{5}-1)}}{2\sqrt{2}}$$

$$=\frac{b\sqrt{\sqrt{(\sqrt{5}+1)^2}-(\sqrt{5}-1)}}{2\sqrt{2}}=\frac{b\sqrt{(\sqrt{5}+1)-(\sqrt{5}-1)}}{2\sqrt{2}}=\frac{b\sqrt{2}}{2\sqrt{2}}=\frac{b}{2}$$

$$\therefore \boldsymbol{Normal\ distance\ of\ each\ face\ from\ the\ centre\ of\ decahedron,}\ \ \boldsymbol{h=\frac{b}{2}}\quad \forall\ \boldsymbol{a=b\sqrt{\sqrt{5}-2}}$$

Above is the **required expression to calculate the normal distance h of each right kite face from the centre of a decahedron. Normal distance h is always equal to inner (inscribed) radius (R_i) i.e. the radius of the spherical surface touching all 10 congruent right kite faces of a decahedron.**

Now, setting the value of a in term of b in the eq(V), we have

$$8{R_o}^2=2a^2+4b^2\quad\Rightarrow {R_o}^2=\frac{a^2+2b^2}{4}$$

$$\Rightarrow {R_o}^2=\frac{\left(b\sqrt{\sqrt{5}-2}\right)^2+2b^2}{4}=\frac{\left(b\sqrt{\sqrt{5}-2}\right)^2+2b^2}{4}=\frac{(\sqrt{5}-2)b^2+2b^2}{4}=\frac{b^2\sqrt{5}}{4}\quad or\quad R_o=\frac{b\sqrt{(\sqrt{5})}}{2}$$

$$\therefore \boldsymbol{Outer\ radius\ of\ decahedron,}\ \ \boldsymbol{R_o=\frac{b(5)^{1/4}}{2}\approx 0.74767439b}\quad \forall\ \boldsymbol{a=b\sqrt{\sqrt{5}-2}}$$

Above is the **required expression to calculate the outer (circumscribed) radius (R_o) of a decahedron having 10 congruent right kite faces.**

Surface Area (A_s) of decahedron: Since, each face of the decahedron is a right kite hence the surface area of the decahedron is given as

$$\boldsymbol{A_s}=10\times(Area\ of\ right\ kite\ face)=10\times\left(2\times(Area\ of\ right\ \Delta ABC)\right)\quad(see\ figure\ 3\ above)$$

$$=20\times\left(\frac{1}{2}ab\right)=10b\left(b\sqrt{\sqrt{5}-2}\right)=10b^2\sqrt{\sqrt{5}-2}$$

$$\therefore\quad \boldsymbol{A_s=10b^2\sqrt{\sqrt{5}-2}\approx 4.858682718b^2}\quad \forall\ \boldsymbol{a=b\sqrt{\sqrt{5}-2}}$$

Volume (V) of decahedron: Since, decahedron has 10 congruent faces each as a right kite hence **a decahedron consists of 10 congruent elementary right pyramids each with right kite base (face)**. Hence the volume (V) of a decahedron is given as (See figure 3 above)

$$V=10\times(volume\ of\ elementary\ right\ pyramid\ with\ right\ kite\ face\ ABCD)$$

$$=10\left\{\frac{1}{3}\times(area\ of\ right\ kite\ face\ ABCD)\times(normal\ height)\right\}=\frac{10}{3}(ab)\times(h)$$

$$=\frac{10b}{3}\left(b\sqrt{\sqrt{5}-2}\right)\frac{b}{2}=\frac{5b^3\sqrt{\sqrt{5}-2}}{3}$$

$$\therefore\ \ \boldsymbol{V=\frac{5b^3\sqrt{\sqrt{5}-2}}{3}\approx 0.809780452b^3 \qquad \forall\ a=b\sqrt{\sqrt{5}-2}}$$

Mean radius $(\boldsymbol{R_m})$: It is the radius of the sphere having a volume equal to that of a given decahedron. It is calculated as follows

$$volume\ of\ sphere\ with\ mean\ radius\ R_m = volume\ of\ given\ decahedron$$

$$\frac{4}{3}\pi(R_m)^3=\frac{5b^3\sqrt{\sqrt{5}-2}}{3}$$

$$\Rightarrow\ (R_m)^3=\frac{5b^3\sqrt{\sqrt{5}-2}}{4\pi}$$

$$\boldsymbol{R_m=b\left(\frac{5\sqrt{\sqrt{5}-2}}{4\pi}\right)^{\frac{1}{3}}\approx 0.578219712b \qquad \forall\ a=b\sqrt{\sqrt{5}-2}}$$

For finite values of a or b $\Rightarrow R_i<R_m<R_o$

Construction of a solid decahedron: In order to construct a solid decahedron having 10 congruent faces each as a right kite with two pairs of unequal sides a & b while one of them is required to be known

Step 1: First we construct right kite base (face) with the help of known values of a & b $\left(\forall\ a=b\sqrt{\sqrt{5}-2}\right)$

Step 2: Construct all its 10 congruent elementary right pyramids with right kite base (face) of a normal height h given as (See figure 3 above)

$$h=\frac{b}{2}\qquad \forall\ \ a=b\sqrt{\sqrt{5}-2}$$

Step 2: paste/bond by joining all 10 elementary right pyramids by overlapping their lateral faces & keeping their apex points coincident with each other such that five right kite faces meet at **each of two identical & diagonally opposite vertices** and three right kite faces meet at each of rest identical 10 vertices. Thus, a solid decahedron, with 10 congruent faces each as a right kite with two pairs of unequal sides of a & b, is obtained.

Conclusions: Let there be any decahedron having 10 congruent faces each as a right kite with two pairs of unequal sides of $\boldsymbol{a}$ & $\boldsymbol{b}$ $\forall$ $\boldsymbol{a=b\sqrt{\sqrt{5}-2}}$ then all its important parameters are determined as tabulated below

Inner (inscribed) radius $(\boldsymbol{R_i})$	$R_i=\frac{b}{2}$
Outer (circumscribed) radius $(\boldsymbol{R_o})$	$R_o=\frac{b(5)^{1/4}}{2}\approx 0.74767439b$

Mean radius $(\boldsymbol{R_m})$	$R_m = b\left(\frac{5\sqrt{\sqrt{5}-2}}{4\pi}\right)^{\frac{1}{3}} \approx 0.578219712b$
Surface area $(\boldsymbol{A_s})$	$A_s = 10b^2\sqrt{\sqrt{5}-2} \approx 4.858682718b^2$
Volume $(\boldsymbol{V})$	$V = \frac{5b^3\sqrt{\sqrt{5}-2}}{3} \approx 0.809780452b^3$

Note: Above articles had been developed *& illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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