Mathematical Analysis of Uniform Tetradecahedron

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Introduction: Here, we are to analyse a **uniform tetradecahedron** having **2 congruent regular hexagonal faces**, **12 congruent trapezoidal faces** & **18 vertices lying on a spherical surface** with a certain radius. Each of 12 trapezoidal faces has three equal sides, two equal **acute** angles each $\boldsymbol{\alpha}\ (\approx \mathbf{79.45^o})$ & two equal **obtuse** angles each $\boldsymbol{\beta}\ (\approx \mathbf{100.55^o})$. (See the figure 1). **The condition, of all 18 vertices lying on a spherical surface**, **governs** & **correlates all the parameters of a uniform tetradecahedron** such as **solid angle** subtended by each face at the centre, **normal distance** of each face from the centre, **outer (circumscribed) radius**, **inner (inscribed) radius**, **mean radius**, **surface area**, **volume** etc. If the length of one of two unequal edges is known then all the dimensions of a uniform tetradecahedron can be easily determined. It is to noted that if the edge length of regular hexagonal face is known then the analysis becomes very easy. We would derive a **mathematical relation** of the side length a of regular hexagonal faces & the radius R of the spherical surface passing through all 18 vertices. Thus, all the dimensions of a uniform tetradecahedron can be easily determined only in terms of edge length a & plane angles, solid angles of each face can also be determined easily.

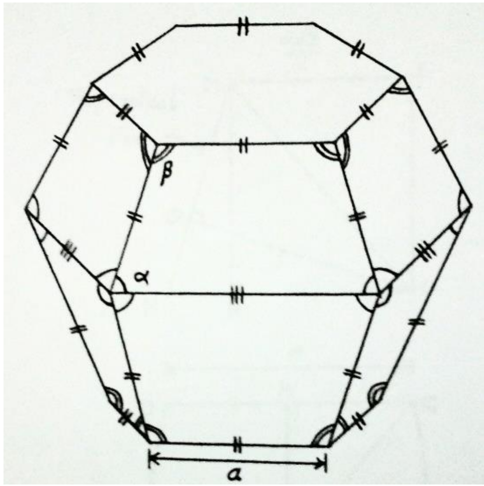

Figure 1: A uniform tetradecahedron has 2 congruent regular hexagonal faces each of edge length a & 12 congruent trapezoidal faces. All its 18 vertices eventually & exactly lie on a spherical surface with a certain radius.

Analysis of Uniform Tetradecahedron: For ease of calculations & understanding, let there be a uniform tetradecahedron, with the centre O, having 2 congruent regular hexagonal faces each with **edge length $\boldsymbol{a}$** & 12 congruent trapezoidal faces and all its 18 vertices lying on a spherical surface with a **radius $\boldsymbol{R_o}$**. Now consider any of 12 congruent trapezoidal faces say ABCD ($AD = BC = CD = a$) & join the vertices A & D to the centre O. (See the figure 2). Join the centre E of the top hexagonal face to the centre O & to the vertex D. Draw a perpendicular DF from the vertex D to the line AO, perpendicular EM from the centre E to the side CD, perpendicular ON from the centre O to the side AB & then join the mid-points M & N of the sides CD & AB respectively in order to obtain trapeziums ADEO & OEMN (See the figure 3 & 4 below). Now we have,

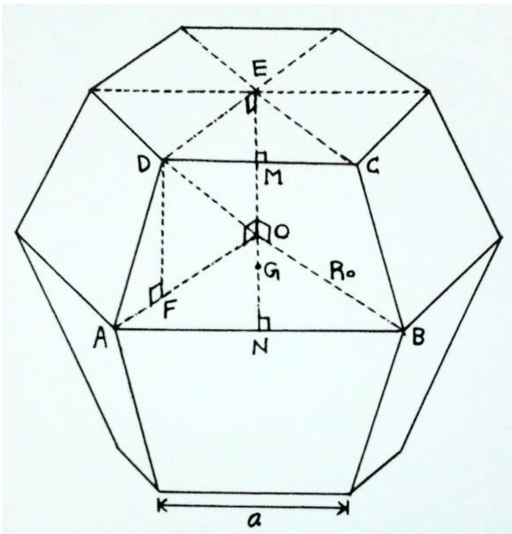

Figure 2: ABCD is one of 12 congruent trapezoidal faces with $AD = BC = CD$. ΔCED & ΔAOB are equilateral triangles. ADEO & OEMN are trapeziums.

$$OA = OD = R_o\ ,\ \ BC = CD = AD = a,\ \ \angle CED = \angle AOB = \frac{360^o}{6} = 60^o$$

Hence, in equilateral triangles ΔCED & ΔAOB , we have

$$\boldsymbol{EC = ED = CD = a}\ \ \&\ \ \boldsymbol{OA = OB = AB = R_o}$$

In right ΔEMD

$$\cos\angle DEM = \frac{EM}{ED} \Rightarrow \cos 30^o = \frac{EM}{a} \ \Rightarrow \boldsymbol{EM = \frac{a\sqrt{3}}{2} = OH}$$

Similarly, in right ΔANO

$$\Rightarrow \boldsymbol{ON = \frac{R_o\sqrt{3}}{2}}$$

In right ΔOED (figure 3)

$$EO = \sqrt{(OD)^2 - (DE)^2} = \sqrt{{R_o}^2 - a^2} \quad \Rightarrow \boldsymbol{EO = DF = \sqrt{{R_o}^2 - a^2}}$$

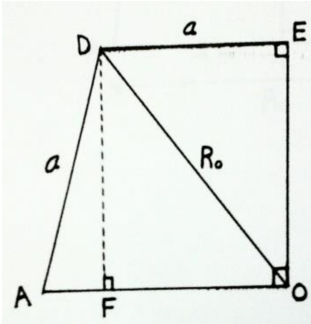

Figure 3: Trapezium ADEO with $\boldsymbol{AD = DE = OF = a,\ OA = OD = R_o}$ & $\boldsymbol{DF = EO}$. The lines DE & AO are parallel.

In right ΔAFD (figure 3)

$$\Rightarrow (AD)^2 = (AF)^2 + (DF)^2 = (OA - OF)^2 + (DF)^2 = (R_o - a)^2 + \left(\sqrt{{R_o}^2 - a^2}\right)^2$$

$$\Rightarrow a^2 = {R_o}^2 + a^2 - 2aR_o + {R_o}^2 + a^2 = 2{R_o}^2 - 2aR_o$$

$$2{R_o}^2 - 2aR_o - a^2 = 0$$

$$\Rightarrow R_o = \frac{2a \pm \sqrt{(-2a)^2 + 8a^2}}{4} = \frac{2a \pm 2a\sqrt{3}}{4} = \frac{a(1 \pm \sqrt{3})}{2}$$

But, $R_o > a > 0$ by taking positive sign, we get

$$\therefore \quad \boldsymbol{R_o = \frac{(1+\sqrt{3})a}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (I)$$

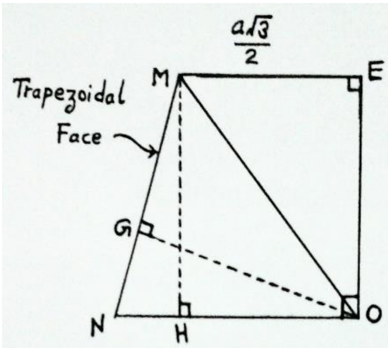

Figure 4: Trapezium OEMN with $\boldsymbol{EM = OH}$ & $\boldsymbol{EO = MH}$. The lines ME & NO are parallel.

Now, draw a perpendicular OG from the centre O to the trapezoidal face ABCD, perpendicular MH from the mid-point M of the side CD to the line ON. Thus in **trapezium OEMN** (See the figure 4), we have

$$MH = EO = \sqrt{{R_o}^2 - a^2} = \sqrt{\left(\frac{(1+\sqrt{3})a}{2}\right)^2 - a^2} = a\sqrt{\frac{(1+3+2\sqrt{3})-4}{4}} = a\sqrt{\frac{2\sqrt{3}}{4}} = a\sqrt{\frac{\sqrt{3}}{2}} = a\sqrt{\sqrt{\frac{3}{4}}}$$

$$\therefore \boldsymbol{MH = EO = a\left(\frac{3}{4}\right)^{\frac{1}{4}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (II)$$

$$\boldsymbol{NH} = ON - OH = ON - EM = \frac{R_o\sqrt{3}}{2} - \frac{a\sqrt{3}}{2} = \frac{\sqrt{3}(R_o - a)}{2} = \frac{\sqrt{3}\left(\frac{(1+\sqrt{3})a}{2} - a\right)}{2} \quad (from\ eq(I))$$

$$= \frac{a\sqrt{3}(1+\sqrt{3}-2)}{4} = \frac{a\sqrt{3}(\sqrt{3}-1)}{4} = \frac{(3-\sqrt{3})a}{4}$$

In right ΔMHN (figure 4)

$$MN = \sqrt{(MH)^2 + (NH)^2} = \sqrt{(EO)^2 + (NH)^2} = \sqrt{\left(a\left(\frac{3}{4}\right)^{\frac{1}{4}}\right)^2 + \left(\frac{(3-\sqrt{3})a}{4}\right)^2}$$

$$= a\sqrt{\frac{\sqrt{3}}{2}+\frac{9+3-6\sqrt{3}}{16}} = a\sqrt{\frac{8\sqrt{3}+12-6\sqrt{3}}{16}} = a\sqrt{\frac{12+2\sqrt{3}}{16}} = a\sqrt{\frac{6+\sqrt{3}}{8}}$$

$$\therefore \quad \boldsymbol{MN = \frac{a}{2}\sqrt{\frac{6+\sqrt{3}}{2}}} \qquad \dots\dots\dots\dots\dots (III)$$

Now, area of ΔOMN can be calculated as follows (from figure 4)

$$area\ of\ \Delta OMN = \frac{1}{2}[(MN)\times(OG)] = \frac{1}{2}[(ON)\times(MH)] \ \Rightarrow\ (MN)\times(OG) = (ON)\times(MH)$$

$$\Rightarrow\ OG = \frac{(ON)\times(MH)}{MN} = \frac{\left(\frac{R_o\sqrt{3}}{2}\right)\times\left(a\left(\frac{3}{4}\right)^{\frac{1}{4}}\right)}{\left(\frac{a}{2}\sqrt{\frac{6+\sqrt{3}}{2}}\right)} = \left(\sqrt{3}\frac{(1+\sqrt{3})a}{2}\right)\sqrt{\frac{2\left(\frac{\sqrt{3}}{2}\right)}{6+\sqrt{3}}} = \frac{(3+\sqrt{3})a}{2}\sqrt{\frac{\sqrt{3}}{6+\sqrt{3}}}$$

$$= \frac{(3+\sqrt{3})a}{2}\sqrt{\frac{1}{2\sqrt{3}+1}} = \frac{(3+\sqrt{3})a}{2}\sqrt{\frac{(2\sqrt{3}-1)}{(2\sqrt{3}+1)(2\sqrt{3}-1)}} = \frac{a}{2}\sqrt{\frac{(2\sqrt{3}-1)(3+\sqrt{3})^2}{(12-1)}}$$

$$= \frac{a}{2}\sqrt{\frac{(2\sqrt{3}-1)(12+6\sqrt{3})}{11}} = \frac{a}{2}\sqrt{\frac{24+18\sqrt{3}}{11}} = a\sqrt{\frac{3(4+3\sqrt{3})}{22}}$$

$$\therefore\ \boldsymbol{OG = a\sqrt{\frac{3(4+3\sqrt{3})}{22}}} \qquad \dots\dots\dots\dots\dots\dots (IV)$$

Normal distance (H_h) of regular hexagonal faces from the centre of uniform tetradecahedron: The normal distance (H_h) of each of 2 congruent regular hexagonal faces from the centre O of a uniform tetradecahedron is given as

$$H_h = EO = a\left(\frac{3}{4}\right)^{\frac{1}{4}} \qquad (from\ the\ eq(II)\ above)$$

$$\therefore\ \boldsymbol{H_h = a\left(\frac{3}{4}\right)^{\frac{1}{4}} \approx 0.930604859a}$$

It's clear that both the congruent regular hexagonal faces are at an equal normal distance H_h from the centre of a uniform tetradecahedron.

Solid angle (ω_h) subtended by each of 2 congruent regular hexagonal faces at the centre of uniform tetradecahedron: We know that the **solid angle (ω) subtended by any regular polygon with each side of length a at any point lying at a distance H on the vertical axis passing through the centre of plane** is given by "**HCR's Theory of Polygon**" as follows

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2+a^2\cot^2\frac{\pi}{n}}}\right)}$$

Hence, by substituting the corresponding values in the above expression, we get the solid angle subtended by each regular hexagonal face at the centre of the uniform tetradecahedron as follows

$$\omega_{hexagon} = 2\pi - 2\times 6\sin^{-1}\left(\frac{2(EO)sin\frac{\pi}{6}}{\sqrt{4(EO)^2 + a^2cot^2\frac{\pi}{6}}}\right) = 2\pi - 12\sin^{-1}\left(\frac{2\left(a\left(\frac{3}{4}\right)^{\frac{1}{4}}\right)\left(\frac{1}{2}\right)}{\sqrt{4\left(a\left(\frac{3}{4}\right)^{\frac{1}{4}}\right)^2 + a^2\left(\sqrt{3}\right)^2}}\right)$$

$$= 2\pi - 12\sin^{-1}\left(\frac{\left(\frac{3}{4}\right)^{\frac{1}{4}}}{\sqrt{4\left(\frac{\sqrt{3}}{2}\right)+3}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{\left(\frac{\sqrt{3}}{2}\right)}{2\sqrt{3}+3}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{1}{2(2+\sqrt{3})}}\right)$$

$$= 2\pi - 12\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{3})}{2(2+\sqrt{3})(2-\sqrt{3})}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{3})}{2(4-3)}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right)$$

$$\boldsymbol{\omega_h = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right) \approx 1.786372116\ sr}$$

Normal distance (H_t) of trapezoidal faces from the centre of uniform tetradecahedron: The normal distance (H_t) of each of 12 congruent trapezoidal faces from the centre of uniform tetradehedron is given as

$$H_t = OG = a\sqrt{\frac{3(4+3\sqrt{3})}{22}} \qquad (from\ the\ eq(IV)\ above)$$

$$\Rightarrow \boldsymbol{H_t = a\sqrt{\frac{3(4+3\sqrt{3})}{22}} \approx 1.119830695a}$$

It's clear that all 12 congruent trapezoidal faces are at an equal normal distance H_t from the centre of a uniform tetradecahedron.

Solid angle (ω_t) subtended by each of 12 congruent trapezoidal faces at the centre of uniform tetradecahedron: Since a uniform tetradecahedron is a closed surface & we know that **the total solid angle, subtended by any closed surface at any point lying inside it, is $4\pi\ sr$ (Ste-radian)** hence the sum of solid angles subtended by **2 congruent regular hexagonal** & **12 congruent trapezoidal faces** at the centre of the uniform tetradecahedron must be $4\pi\ sr$. Thus we have

$$2[\omega_{hexagon}] + 12[\omega_{trapezium}] = 4\pi \ \ or\ \ 12[\omega_{trapezium}] = 4\pi - 2[\omega_{hexagon}]$$

$$\omega_{trapezium} = \frac{2\pi - \omega_{hexagon}}{6} = \frac{2\pi - \left[2\pi - 12\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right)\right]}{6} = 2\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right)$$

$$\therefore \ \boldsymbol{\omega_t = 2\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right) \approx 0.749468865\ sr}$$

It's clear from the above results that the solid angle subtended by each of 2 regular hexagonal faces is greater than the solid angle subtended by each of 12 trapezoidal faces at the centre of a uniform tetradecahedron.

It's also clear from the above results that $\boldsymbol{H_t > H_h}$ i.e. the normal distance (H_t) of trapezoidal faces is greater than the normal distance H_h of the regular hexagonal faces from the centre of a uniform tetradecahedron i.e. regular hexagonal **faces are closer to the centre as compared to the trapezoidal faces in a uniform tetradecahedron**.

Interior angles $(\boldsymbol{\alpha}\ \&\ \boldsymbol{\beta})$ of the trapezoidal faces of uniform tetradecahedron: From above figures 1 & 2, let α be acute angle & β be obtuse angle. Acute angle α is determined as follows

$$\sin\angle BAD = \frac{MN}{AD} \Rightarrow sin\alpha = \frac{\left(\frac{a}{2}\sqrt{\frac{6+\sqrt{3}}{2}}\right)}{a} \quad from\ eq(III) above$$

$$= \frac{1}{2}\sqrt{\frac{6+\sqrt{3}}{2}} \quad or\ \alpha = \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{6+\sqrt{3}}{2}}\right)$$

$$\Rightarrow \boldsymbol{Acute\ angle,\ \ \alpha = \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{6+\sqrt{3}}{2}}\right) \approx 79.45470941^o \approx 79^o27'16.95''}$$

In trapezoidal face ABCD, we know that the **sum of all interior angles (of a quadrilateral) is $\boldsymbol{360^o}$**

$$\therefore 2\alpha + 2\beta = 360^o \ \ or\ \ \beta = 180^o - \alpha$$

$$\therefore \boldsymbol{Obtuse\ angle,\ \ \beta = 180^o - \alpha \approx 100.5452906^o \approx 100^o32'43.05''}$$

Sides of the trapezoidal face of uniform tetradecahedron: All the sides of each trapezoidal face can be determined as follows (See figure (3) & (5) above)

$$\boldsymbol{CD = AD = BC = a\ \&\ AB = R_o = \frac{(1+\sqrt{3})a}{2}} \quad (from\ eq(I)\ above)$$

$$\boldsymbol{Distance\ between\ parallel\ sides\ AB\ \&\ CD\ of\ trapezoidal\ face\ ABCD}$$

$$\therefore \ \boldsymbol{MN = \frac{a}{2}\sqrt{\frac{6+\sqrt{3}}{2}}} \qquad (from\ eq(III) above)$$

Hence, the **area of each of 12 congruent trapezoidal faces of a uniform tetradecahedron** is given as follows

$$\boldsymbol{Area\ of\ trapezium\ ABCD} = \frac{1}{2}(sum\ of\ parallel\ sides) \times (normal\ distance\ between\ parallel\ sides)$$

$$= \frac{1}{2}(AB+CD)(MN) = \frac{1}{2}(R_o + a)\left(\frac{a}{2}\sqrt{\frac{6+\sqrt{3}}{2}}\right) = \frac{a}{4}\left(\frac{(1+\sqrt{3})a}{2} + a\right)\sqrt{\frac{6+\sqrt{3}}{2}}$$

$$=\frac{a^2}{8}(1+\sqrt{3}+2)\sqrt{\frac{6+\sqrt{3}}{2}}=\frac{a^2}{8}(3+\sqrt{3})\sqrt{\frac{6+\sqrt{3}}{2}}=\frac{a^2}{8}\sqrt{\frac{(6+\sqrt{3})(3+\sqrt{3})^2}{2}}=\frac{a^2}{8}\sqrt{\frac{(6+\sqrt{3})(12+6\sqrt{3})}{2}}$$

$$=\frac{a^2}{8}\sqrt{\frac{(6+\sqrt{3})(12+6\sqrt{3})}{2}}=\frac{a^2}{8}\sqrt{\frac{90+48\sqrt{3}}{2}}=\frac{a^2}{8}\sqrt{45+24\sqrt{3}}$$

$$=\frac{a^2}{8}\sqrt{45+24\sqrt{3}}\approx 1.163032266a^2$$

Important parameters of a uniform tetradecahedron:

1. **Inner (inscribed) radius (R_i):** It is the radius of the largest sphere inscribed (trapped inside) by a uniform tetradecahedron. The largest inscribed sphere always touches both the congruent regular hexagonal faces but does not touch any of 12 congruent trapezoidal faces at all since both the hexagonal faces are closer to the centre as compared to all 12 trapezoidal faces. Thus, inner radius is always equal to the normal distance (H_h) of the regular hexagonal faces from the centre of a uniform tetradecahedron & is given as follows

$$R_i=a\left(\frac{3}{4}\right)^{\frac{1}{4}}\approx 0.930604859a$$

Hence, the **volume of inscribed sphere** is given as

$$V_{inscribed}=\frac{4}{3}\pi(R_i)^3=\frac{4}{3}\pi\left(a\left(\frac{3}{4}\right)^{\frac{1}{4}}\right)^3\approx 3.375861004a^3$$

2. **Outer (circumscribed) radius (R_o):** It is the radius of the smallest sphere circumscribing a uniform tetradecahedron or it's the radius of a spherical surface passing through all 18 vertices of a uniform tetradecahedron. It is given as follows

$$R_o=\frac{(1+\sqrt{3})a}{2}\approx 1.366025404a$$

Hence, the **volume of circumscribed sphere** is given as

$$V_{circumscribed}=\frac{4}{3}\pi(R_o)^3=\frac{4}{3}\pi\left(\frac{(1+\sqrt{3})a}{2}\right)^3=10.67738585a^3$$

3. **Surface area (A_s):** We know that a uniform tetradecahedron has 2 congruent regular hexagonal faces & 12 congruent trapezoidal faces. Hence, its surface area is given as follows

$$A_s=2(area\ of\ regular\ hexagon)+12(area\ of\ trapezium\ ABCD)\qquad (see\ figure\ 2\ above)$$

We know that **area of any regular n-polygon** with each side of length a is given as

$$A=\frac{1}{4}na^2\cot\frac{\pi}{n}\quad (for\ regular\ hexagon,\ \ n=6)$$

Hence, by substituting all the corresponding values in the above expression, we get

$$A_s = 2 \times \left(\frac{1}{4} \times 6a^2 cot\frac{\pi}{6}\right) + 12 \times \left(\frac{1}{2}(AB + CD)(MN)\right) = 3a^2\sqrt{3} + 12 \times \left(\frac{a^2}{8}\sqrt{45 + 24\sqrt{3}}\right)$$

$$= 3a^2\sqrt{3} + \frac{3a^2}{2}\sqrt{45 + 24\sqrt{3}} = \frac{3a^2}{2}\left(2\sqrt{3} + \sqrt{45 + 24\sqrt{3}}\right)$$

$$\therefore \ \boldsymbol{A_s = \frac{3a^2}{2}\left(2\sqrt{3} + \sqrt{45 + 24\sqrt{3}}\right) \approx 19.15253962a^2}$$

4. **Volume (V):** We know that a uniform tetradecahedron has 2 congruent regular hexagonal & 12 congruent trapezoidal faces. Hence, the volume (V) of the uniform tetradecahedron is the sum of volumes of all its elementary right pyramids with regular hexagonal & trapezoidal bases (faces) given as follows

$$V = 2(volume\ of\ right\ pyramid\ with\ regular\ hexagonal\ base) + 12(volume\ of\ right\ pyramid\ with\ trapezoidal\ base\ ABCD)$$

$$= 2\left(\frac{1}{3}(area\ of\ regular\ hexagon) \times H_h\right) + 12\left(\frac{1}{3}(area\ of\ trapezium\ ABCD) \times H_t\right)$$

$$= 2\left(\frac{1}{3}\left(\frac{1}{4} \times 6a^2 cot\frac{\pi}{6}\right) \times a\left(\frac{3}{4}\right)^{\frac{1}{4}}\right) + 12\left(\frac{1}{3}\left(\frac{a^2}{8}\sqrt{45 + 24\sqrt{3}}\right) \times a\sqrt{\frac{3(4 + 3\sqrt{3})}{22}}\right)$$

$$= a^3(\sqrt{3})\left(\frac{3}{4}\right)^{\frac{1}{4}} + \frac{a^3}{2}\sqrt{45 + 24\sqrt{3}}\sqrt{\frac{3(4 + 3\sqrt{3})}{22}} = a^3\sqrt{\frac{3\sqrt{3}}{2}} + \frac{a^3}{2}\sqrt{\frac{3(4 + 3\sqrt{3})(45 + 24\sqrt{3})}{22}}$$

$$= a^3\sqrt{\frac{3\sqrt{3}}{2}} + \frac{a^3}{2}\sqrt{\frac{3(396 + 231\sqrt{3})}{22}} = a^3\sqrt{\frac{3\sqrt{3}}{2}} + \frac{a^3}{2}\sqrt{\frac{3(396 + 231\sqrt{3})}{22}}$$

$$= a^3\sqrt{\frac{3\sqrt{3}}{2}} + \frac{a^3}{2}\sqrt{\frac{99(12 + 7\sqrt{3})}{22}} = a^3\left(\sqrt{\frac{3\sqrt{3}}{2}} + \frac{3}{2}\sqrt{\frac{12 + 7\sqrt{3}}{2}}\right) = a^3\left(\frac{2\sqrt{6\sqrt{3}} + 3\sqrt{24 + 14\sqrt{3}}}{4}\right)$$

$$\therefore \ \boldsymbol{V = a^3\left(\frac{2\sqrt{6\sqrt{3}} + 3\sqrt{24 + 14\sqrt{3}}}{4}\right) \approx 6.821451822a^3}$$

5. **Mean radius (R_m):** It is the radius of the sphere having a volume equal to that of a uniform tetradecahedron. It is calculated as follows

$$volume\ of\ sphere\ with\ mean\ radius\ R_m = volume\ of\ the\ uniform\ tetradecahedron$$

$$\frac{4}{3}\pi(R_m)^3 = a^3\left(\frac{2\sqrt{6\sqrt{3}} + 3\sqrt{24 + 14\sqrt{3}}}{4}\right) \ \Rightarrow \ (R_m)^3 = \left(\frac{6\sqrt{6\sqrt{3}} + 9\sqrt{24 + 14\sqrt{3}}}{16\pi}\right)a^3$$

$$\therefore \ \boldsymbol{R_m = a\left(\frac{6\sqrt{6\sqrt{3}} + 9\sqrt{24 + 14\sqrt{3}}}{16\pi}\right)^{\frac{1}{3}} \approx 1.176511208a}$$

It's clear from above results that $\boldsymbol{R_i < R_m < R_o}$

Construction of a solid uniform tetradecahedron: In order to construct a there are two methods

1. Construction from elementary right pyramids: In this method, first we construct all elementary right pyramids as follows

Construct 2 congruent right pyramids with regular hexagonal base of side length a & normal height (H_h)

$$H_h = a\left(\frac{3}{4}\right)^{\frac{1}{4}} \approx 0.930604859a$$

Construct 12 congruent right pyramids with **trapezoidal base ABCD** of sides $AB,\ BC = AD = CD$ & normal height (H_t)

$$H_t = a\sqrt{\frac{3(4+3\sqrt{3})}{22}} \approx 1.119830695a$$

$$AD = BC = CD = a \ \ \& \ \ AB = \frac{(1+\sqrt{3})a}{2} \approx 1.366025404a \qquad (See\ figure\ 2\ above)$$

$$Acute\ angle, \alpha \approx 79.45470941^o$$

$$AB\ \&\ CD\ are\ parallel\ sides\ \&\ AD\ \&\ AC\ are\ equal\ but\ non\ parallel\ sides$$

Now, paste/bond by joining all these elementary right pyramids by overlapping their lateral surfaces & keeping their apex points coincident with each other such that all 6 edges of each regular hexagonal base (face) coincide with the edges of 6 trapezoidal bases (faces). Thus a solid uniform tetradecahedron, with 2 congruent regular hexagonal faces, 12 congruent trapezoidal faces & 18 vertices lying on a spherical surface, is obtained.

2. Facing a solid sphere: It is a method of facing, first we select a **blank as a solid sphere** of certain material (i.e. metal, alloy, composite material etc.) & with suitable diameter in order to obtain the maximum desired edge length of the hexagonal face of a uniform tetradecahedron. Then, we perform the facing operations on the solid sphere to generate 2 congruent regular hexagonal faces each with edge length a & 12 congruent trapezoidal faces.

Let there be a blank as a solid sphere with a diameter D. Then the edge length a, of each regular hexagonal face of a uniform tetradecahedron of the maximum volume to be produced, can be co-related with the diameter D by **relation of outer radius $(\boldsymbol{R_o})$ with edge length $(\boldsymbol{a})$ of the hexagonal face** as follows

$$R_o = \frac{(1+\sqrt{3})a}{2}$$

Now, substituting $R_o = {}^{D}/_{2}$ in the above expression, we have

$$\frac{D}{2} = \frac{(1+\sqrt{3})a}{2} \quad or \quad a = \frac{D}{(\sqrt{3}+1)} = \frac{D(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{D(\sqrt{3}-1)}{2}$$

$$\boldsymbol{a = \frac{D(\sqrt{3}-1)}{2} \approx 0.366025403D}$$

Above relation is very useful for determining the edge length a of regular hexagonal face of a uniform tetradecahedron to be produced from a solid sphere with known diameter D for manufacturing purpose.

Hence, the **maximum volume of uniform tetradecahedron** produced from a solid sphere is given as follows

$$V_{max} = a^3\left(\frac{2\sqrt{6\sqrt{3}}+3\sqrt{24+14\sqrt{3}}}{4}\right) = \left(\frac{D(\sqrt{3}-1)}{2}\right)^3\left(\frac{2\sqrt{6\sqrt{3}}+3\sqrt{24+14\sqrt{3}}}{4}\right)$$

$$= D^3(6\sqrt{3}-10)\left(\frac{2\sqrt{6\sqrt{3}}+3\sqrt{24+14\sqrt{3}}}{32}\right) = D^3(3\sqrt{3}-5)\left(\frac{2\sqrt{6\sqrt{3}}+3\sqrt{24+14\sqrt{3}}}{16}\right)$$

$$= D^3\left(\frac{2\sqrt{6\sqrt{3}(3\sqrt{3}-5)^2}+3\sqrt{(24+14\sqrt{3})(3\sqrt{3}-5)^2}}{16}\right) = D^3\left(\frac{2\sqrt{3(26\sqrt{3}-45)}+3\sqrt{(2\sqrt{3}-3)}}{8}\right)$$

$$\boldsymbol{V_{max} = D^3\left(\frac{2\sqrt{3(26\sqrt{3}-45)}+3\sqrt{2\sqrt{3}-3}}{8}\right) \approx 0.334511075D^3}$$

M**inimum volume of material removed** is given as

$$(V_{removed})_{min} = (volume\ of\ parent\ sphere\ with\ diameter\ D) - (volume\ of\ uniform\ tetradecahedron)$$

$$= \frac{\pi}{6}D^3 - D^3\left(\frac{2\sqrt{3(26\sqrt{3}-45)}+3\sqrt{2\sqrt{3}-3}}{8}\right) = \left(\frac{\pi}{6}-\frac{2\sqrt{3(26\sqrt{3}-45)}+3\sqrt{2\sqrt{3}-3}}{8}\right)D^3$$

$$\boldsymbol{(V_{removed})_{min} = \left(\frac{\pi}{6}-\frac{2\sqrt{3(26\sqrt{3}-45)}+3\sqrt{2\sqrt{3}-3}}{8}\right)D^3 \approx 0.1890877D^3}$$

Percentage $(\%)$ **of minimum volume of material removed**

$$\%\ \boldsymbol{of}\ \boldsymbol{V_{removed}} = \frac{minimum\ volume\ removed}{total\ volume\ of\ sphere} \times 100$$

$$= \frac{\left(\frac{\pi}{6}-\frac{2\sqrt{3(26\sqrt{3}-45)}+3\sqrt{2\sqrt{3}-3}}{8}\right)D^3}{\frac{\pi}{6}D^3} \times 100 = \boldsymbol{\left(1-\frac{6\sqrt{3(26\sqrt{3}-45)}+9\sqrt{2\sqrt{3}-3}}{4\pi}\right) \times 100}$$

$$\boldsymbol{\approx 36.11\%}$$

It's obvious that when a solid uniform tetradecahedron of the maximum volume is produced from a solid sphere then about 36.11% of material is removed as scraps. Thus, we can select optimum diameter of blank as a solid sphere to produce a solid uniform tetradecahedron of the maximum volume (or with the maximum desired edge length a of regular hexagonal face)

Conclusions: Let there be a uniform tetradecahedron having 2 congruent regular hexagonal faces each with edge length a, 12 congruent trapezoidal faces & 18 vertices lying on a spherical surface with certain radius then all its important parameters are calculated/determined as tabulated below

Congruent polygonal faces	No. of faces	Normal distance of each face from the centre of the uniform tetradecahedron	Solid angle subtended by each face at the centre of the uniform tetradecahedron
Regular hexagon	2	$a\left(\frac{3}{4}\right)^{\frac{1}{4}} \approx 0.930604859a$	$2\pi - 12\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right) \approx 1.786372116\ sr$
Trapezium	12	$a\sqrt{\frac{3(4+3\sqrt{3})}{22}} \approx 1.119830695a$	$2\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{2}}\right) \approx 0.749468865\ sr$

Inner (inscribed) radius (R_i)	$R_i = a\left(\frac{3}{4}\right)^{\frac{1}{4}} \approx 0.930604859a$
Outer (circumscribed) radius (R_o)	$R_o = \frac{(1+\sqrt{3})a}{2} \approx 1.366025404a$
Mean radius (R_m)	$R_m = a\left(\frac{6\sqrt{6\sqrt{3}} + 9\sqrt{24+14\sqrt{3}}}{16\pi}\right)^{\frac{1}{3}} \approx 1.176511208a$
Surface area (A_s)	$A_s = \frac{3a^2}{2}\left(2\sqrt{3} + \sqrt{45+24\sqrt{3}}\right) \approx 19.15253962a^2$
Volume (V)	$V = a^3\left(\frac{2\sqrt{6\sqrt{3}} + 3\sqrt{24+14\sqrt{3}}}{4}\right) \approx 6.821451822a^3$

Note: Above articles had been developed & *illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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