$Mathematical\ Analysis\ of\ Great\ Rhombicosidodecahedron$

$Application\ of\ HCR's\ Theory\ of\ Polygon$

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Introduction: A great rhombicosidodecahedron is the largest Archimedean solid which has 30 congruent square faces, 20 congruent regular hexagonal faces & 12 congruent regular decagonal faces each having equal edge length. It has 180 edges & 120 vertices lying on a spherical surface with a certain radius. It is created/generated by expanding a truncated dodecahedron having 20 equilateral triangular faces & 12 regular decagonal faces. Thus by the expansion, each of 30 originally truncated edges changes into a square face, each of 20 triangular faces of the original solid changes into a regular hexagonal face & 12 regular decagonal faces of the original solid remain unchanged i.e. decagonal faces are shifted radially. Thus a solid with 30 squares, 20 hexagonal & 12 decagonal faces, is obtained which is called **great rhombicosidodecahedron** which is **the largest Archimedean solid**. (See figure 1), thus we have

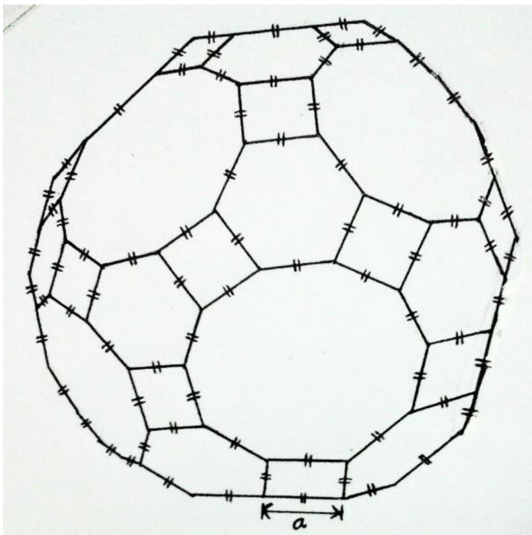

Figure 1: A great rhombicosidodecahedron having 30 congruent square faces, 20 congruent regular hexagonal faces & 12 congruent regular decagonal faces each of equal edge length a

$no.of\ square\ faces = 30,\ no.of\ regular\ hexagonal\ faces = 20$

$no.of\ regular\ decagonal\ faces = 12,$

$no.of\ edges =$
$12(no.of\ edges\ in\ a\ decagonal\ face) + 2(no.of\ square\ faces) =$
$12(10) + 2(30) = 120 + 60 = 180$

$no.of\ vertices = 12(no.of\ vertices\ in\ a\ decagonal\ face) = 12(10) = 120$

We would apply HCR's Theory of Polygon to derive a mathematical relationship between radius R of the spherical surface passing through all 120 vertices & the edge length a of a great rhombicosidodecahedron for calculating its important parameters such as normal distance of each face, surface area, volume etc.

Derivation of outer (circumscribed) radius (R_o) of great rhombicosidodecahedron:

Let R_o be the radius of the spherical surface passing through all 120 vertices of a great rhombicosidodecahedron with edge length a & the centre O. Consider a square face ABCD with centre O_1, regular hexagonal face EFGHIJ with centre O_2 & regular decagonal face KLMNPQRSTU with centre O_3 (see the figure 2 below)

Normal distance (H_s) of square face ABCD from the centre O of the great rhombicosidodecahedron is calculated as follows

In right ΔAO_1O (figure 2)

$$OO_1 = \sqrt{(OA)^2 - (O_1A)^2} \qquad \left(\boldsymbol{O_1A = \frac{a}{\sqrt{2}}} = circumscribed\ radius\ of\ square\right)$$

$$\Rightarrow \boldsymbol{H_s} = \sqrt{(R_o)^2 - \left(\frac{a}{\sqrt{2}}\right)^2} = \sqrt{\frac{\boldsymbol{2R_o}^2 - \boldsymbol{a}^2}{\boldsymbol{2}}} \qquad \dots\dots\dots\dots.(I)$$

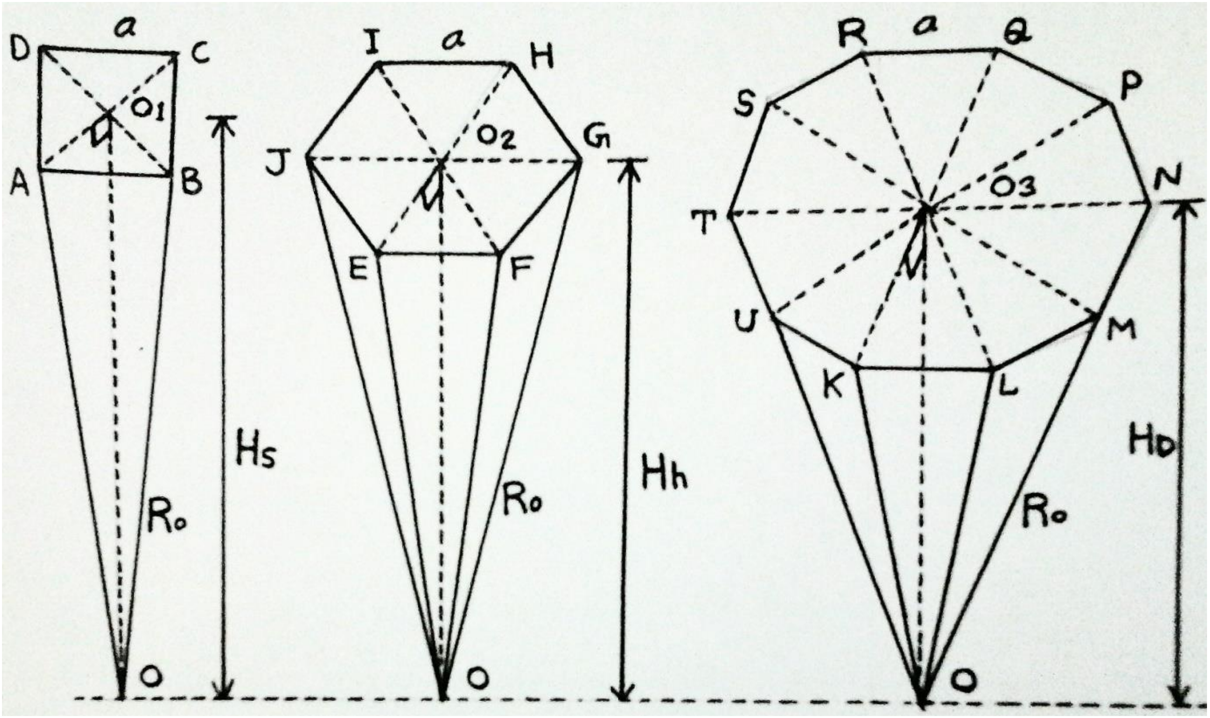

Figure 2: A square face ABCD, regular hexagonal face EFGHIJ & regular decagonal face KLMNPQRSTU each of edge length a

We know that the solid angle (ω) subtended by any regular polygon with each side of length a at any point lying at a distance H on the vertical axis passing through the centre of plane is given by "**HCR's Theory of Polygon**" as follows

$$\omega = 2\pi - 2n\sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2 + a^2\cot^2\frac{\pi}{n}}}\right)$$

Hence, by substituting the corresponding values in the above expression, we get the solid angle $(\boldsymbol{\omega_s})$ subtended by each square face (ABCD) at the centre of the great rhombicosidodecahedron as follows

$$\omega_s = 2\pi - 2\times 4\sin^{-1}\left(\frac{2\left(\sqrt{\frac{2{R_o}^2 - a^2}{2}}\right)\sin\frac{\pi}{4}}{\sqrt{4\left(\sqrt{\frac{2{R_o}^2 - a^2}{2}}\right)^2 + a^2\cot^2\frac{\pi}{4}}}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\frac{\sqrt{2}\sqrt{2{R_o}^2 - a^2}\times\frac{1}{\sqrt{2}}}{\sqrt{4{R_o}^2 - 2a^2 + a^2(1)^2}}\right) = 2\pi - 8\sin^{-1}\left(\frac{\sqrt{2{R_o}^2 - a^2}}{\sqrt{4{R_o}^2 - a^2}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2{R_o}^2 - a^2}{4{R_o}^2 - a^2}}\right)$$

$$\boldsymbol{Let\ \frac{R_o}{a} = x > 1\ (any\ arbitrary\ variable)}$$

$$\Rightarrow \boldsymbol{\omega_s} = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2\left(\frac{R_o}{a}\right)^2 - 1}{4\left(\frac{R_o}{a}\right)^2 - 1}}\right) = \boldsymbol{2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2 - 1}{4x^2 - 1}}\right)} \quad \ldots\ldots\ldots\ldots(II)$$

Similarly, **Normal distance** ($\boldsymbol{H_h}$) of **regular hexagonal face** EFGHIJ from the centre O of the great rhombicosidodecahedron is calculated as follows

In right ΔEO_2O (figure 2)

$$OO_2 = \sqrt{(OE)^2 - (O_2E)^2} \qquad\qquad (\boldsymbol{O_2E} = \boldsymbol{a} = circumscribed\ radius\ of\ regular\ hexagon)$$

$$\Rightarrow \boldsymbol{H_h} = \sqrt{(R_o)^2 - (a)^2} = \sqrt{\boldsymbol{R_o}^2 - \boldsymbol{a}^2} \qquad \ldots\ldots\ldots\ldots\ldots(III)$$

Hence, by substituting all the corresponding values, the solid angle ($\boldsymbol{\omega_h}$) subtended by each regular hexagonal face (EFGHIJ) at the centre of the great rhombicosidodecahedron is given as follows

$$\omega_h = 2\pi - 2 \times 6 \sin^{-1}\left(\frac{2\left(\sqrt{R_o{}^2 - a^2}\right) sin\frac{\pi}{6}}{\sqrt{4\left(\sqrt{R_o{}^2 - a^2}\right)^2 + a^2 cot^2\frac{\pi}{6}}}\right)$$

$$= 2\pi - 12 \sin^{-1}\left(\frac{2\sqrt{R_o{}^2 - a^2} \times \frac{1}{2}}{\sqrt{4R_o{}^2 - 4a^2 + a^2(\sqrt{3})^2}}\right) = 2\pi - 12 \sin^{-1}\left(\frac{\sqrt{R_o{}^2 - a^2}}{\sqrt{4R_o{}^2 - a^2}}\right)$$

$$\Rightarrow \boldsymbol{\omega_h} = 2\pi - 12 \sin^{-1}\left(\sqrt{\frac{R_o{}^2 - a^2}{4R_o{}^2 - a^2}}\right) = \boldsymbol{2\pi - 12 \sin^{-1}\left(\sqrt{\frac{x^2 - 1}{4x^2 - 1}}\right)} \qquad \ldots\ldots\ldots\ldots\ldots(IV)$$

Similarly, **Normal distance** ($\boldsymbol{H_D}$) of **regular decagonal face** KLMNPQRSTU from the centre O of the great rhombicosidodecahedron is calculated as follows

In right ΔKO_3O (figure 2)

$$\Rightarrow OO_3 = \sqrt{(OE)^2 - (O_3K)^2}$$

$$\boldsymbol{O_3K} = circumscribed\ radius\ of\ regular\ decagon = \frac{\boldsymbol{a}}{\boldsymbol{2}}\boldsymbol{cosec 18^o} = \frac{\boldsymbol{a}}{\boldsymbol{2}}(\boldsymbol{1} + \sqrt{\boldsymbol{5}})$$

$$\Rightarrow \boldsymbol{H_D} = \sqrt{(R_o)^2 - \left(\frac{a}{2}(1 + \sqrt{5})\right)^2} = \sqrt{(R_o)^2 - \frac{a^2}{4}(6 + 2\sqrt{5})} = \sqrt{\frac{\boldsymbol{2R_o}^2 - (\boldsymbol{3} + \sqrt{\boldsymbol{5}})\boldsymbol{a}^2}{\boldsymbol{2}}} \qquad \ldots\ldots\ldots\ldots\ldots(V)$$

Hence, by substituting all the corresponding values, the solid angle ($\boldsymbol{\omega_D}$) subtended by each regular decagonal face (KLMNPQRSTU) at the centre of great rhombicosidodecahedron is given as follows

$$\omega_D = 2\pi - 2 \times 10 \sin^{-1}\left(\frac{2\left(\sqrt{\frac{2R_o{}^2 - (3 + \sqrt{5})a^2}{2}}\right) sin\frac{\pi}{10}}{\sqrt{4\left(\sqrt{\frac{2R_o{}^2 - (3 + \sqrt{5})a^2}{2}}\right)^2 + a^2 cot^2\frac{\pi}{10}}}\right)$$

$$= 2\pi - 20\sin^{-1}\left(\frac{\sqrt{4{R_o}^2 - 2(3+\sqrt{5})a^2}\left(\frac{\sqrt{5}-1}{4}\right)}{\sqrt{4{R_o}^2 - 2(3+\sqrt{5})a^2 + a^2\left(\sqrt{5+2\sqrt{5}}\right)^2}}\right)$$

$$= 2\pi - 20\sin^{-1}\left(\frac{\sqrt{4\left(\frac{6-2\sqrt{5}}{16}\right){R_o}^2 - 2(3+\sqrt{5})\left(\frac{6-2\sqrt{5}}{16}\right)a^2}}{\sqrt{4{R_o}^2 - 2(3+\sqrt{5})a^2 + (5+2\sqrt{5})a^2}}\right)$$

$$= 2\pi - 20\sin^{-1}\left(\frac{\sqrt{\left(\frac{3-\sqrt{5}}{2}\right){R_o}^2 - a^2}}{\sqrt{4{R_o}^2 - a^2}}\right)$$

$$\boldsymbol{\omega_D} = 2\pi - 20\sin^{-1}\left(\sqrt{\frac{\left(\frac{3-\sqrt{5}}{2}\right){R_o}^2 - a^2}{4{R_o}^2 - a^2}}\right) = \mathbf{2\pi - 20\sin^{-1}}\left(\sqrt{\frac{\left(\frac{\mathbf{3}-\sqrt{\mathbf{5}}}{\mathbf{2}}\right)\boldsymbol{x}^{\mathbf{2}} - \mathbf{1}}{\mathbf{4}\boldsymbol{x}^{\mathbf{2}} - \mathbf{1}}}\right) \quad \ldots\ldots\ldots\ldots\ldots (VI)$$

Since a great rhombicosidodecahedron is a **closed surface** & we know that **the total solid angle, subtended by any closed surface at any point lying inside it, is $4\pi\ sr$ (Ste-radian)** hence the sum of solid angles subtended by **30 congruent square faces, 20 congruent regular hexagonal faces & 12 congruent regular decagonal faces** at the centre of the great rhombicosidodecahedron must be $4\pi\ sr$. Thus we have

$$\mathbf{30}[\boldsymbol{\omega_s}] + \mathbf{20}[\boldsymbol{\omega_h}] + \mathbf{12}[\boldsymbol{\omega_D}] = \mathbf{4\pi}$$

Now, by substituting the values of $\boldsymbol{\omega_s}$, $\boldsymbol{\omega_h}$ & $\boldsymbol{\omega_D}$ from eq(II), (IV) & (VI) in the above expression we get

$$30\left[2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right)\right] + 20\left[2\pi - 12\sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right)\right] + 12\left[2\pi - 20\sin^{-1}\left(\sqrt{\frac{\left(\frac{3-\sqrt{5}}{2}\right)x^2 - 1}{4x^2-1}}\right)\right] = 4\pi$$

$$\Rightarrow 240\left[\sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{\left(\frac{3-\sqrt{5}}{2}\right)x^2 - 1}{4x^2-1}}\right)\right] = 124\pi - 4\pi = 120\pi$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{\left(\frac{3-\sqrt{5}}{2}\right)x^2 - 1}{4x^2-1}}\right) = \frac{120\pi}{240} = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right)+\sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right)=\frac{\pi}{2}-\sin^{-1}\left(\sqrt{\frac{\left(\frac{3-\sqrt{5}}{2}\right)x^2-1}{4x^2-1}}\right)$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\sqrt{1-\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right)^2}+\sqrt{\frac{x^2-1}{4x^2-1}}\sqrt{1-\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right)^2}\right)=\cos^{-1}\left(\sqrt{\frac{\left(\frac{3-\sqrt{5}}{2}\right)x^2-1}{4x^2-1}}\right)$$

$$\left(\boldsymbol{since},\quad \boldsymbol{\sin^{-1}X+\sin^{-1}Y=\sin^{-1}\left(X\sqrt{1-Y^2}+Y\sqrt{1-X^2}\right)}\quad \& \quad \boldsymbol{\frac{\pi}{2}-\sin^{-1}X=\cos^{-1}X}\right)$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\sqrt{\frac{3x^2}{4x^2-1}}+\sqrt{\frac{x^2-1}{4x^2-1}}\sqrt{\frac{2x^2}{4x^2-1}}\right)=\sin^{-1}\left(\sqrt{1-\left(\sqrt{\frac{\left(\frac{3-\sqrt{5}}{2}\right)x^2-1}{4x^2-1}}\right)^2}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{x\sqrt{3(2x^2-1)}}{4x^2-1}+\frac{x\sqrt{2(x^2-1)}}{4x^2-1}\right)=\sin^{-1}\left(\sqrt{\frac{\left(\frac{5+\sqrt{5}}{2}\right)x^2}{4x^2-1}}\right)=\sin^{-1}\left(x\sqrt{\frac{(5+\sqrt{5})}{2(4x^2-1)}}\right)$$

$$\Rightarrow \frac{x\sqrt{3(2x^2-1)}}{4x^2-1}+\frac{x\sqrt{2(x^2-1)}}{4x^2-1}=x\sqrt{\frac{(5+\sqrt{5})}{2(4x^2-1)}}$$

$$\Rightarrow \sqrt{2}\sqrt{3(2x^2-1)}+\sqrt{2}\sqrt{2(x^2-1)}=\sqrt{(5+\sqrt{5})(4x^2-1)}$$

$$\Rightarrow \left(\sqrt{6(2x^2-1)}+2\sqrt{(x^2-1)}\right)^2=\left(\sqrt{(5+\sqrt{5})(4x^2-1)}\right)^2$$

$$\Rightarrow 6(2x^2-1)+4(x^2-1)+4\sqrt{6(x^2-1)(2x^2-1)}=(5+\sqrt{5})(4x^2-1)$$

$$\Rightarrow 4(5+\sqrt{5})x^2-(5+\sqrt{5})-16x^2+10=4\sqrt{6(x^2-1)(2x^2-1)}$$

$$\Rightarrow \left(4(1+\sqrt{5})x^2+(5-\sqrt{5})\right)^2=\left(4\sqrt{6(x^2-1)(2x^2-1)}\right)^2$$

$$\Rightarrow 32(3+\sqrt{5})x^4+32\sqrt{5}\,x^2+10(3-\sqrt{5})=96(x^2-1)(2x^2-1)=192x^4-288x^2+96$$

$$\Rightarrow (96-32\sqrt{5})x^4-(288+32\sqrt{5})\,x^2+(66+10\sqrt{5})=0$$

$$\Rightarrow 32\,(3-\sqrt{5})x^4-32(9+\sqrt{5})\,x^2+2(33+5\sqrt{5})=0$$

Now, solving the above **biquadratic equation for the values of $x>1$** as follows

$$\Rightarrow x^2 = \frac{-\left(-32\left(9+\sqrt{5}\right)\right) \pm \sqrt{\left(-32\left(9+\sqrt{5}\right)\right)^2 - 4\left(32\left(3-\sqrt{5}\right)\right)\left(2\left(33+5\sqrt{5}\right)\right)}}{2 \times 32\left(3-\sqrt{5}\right)}$$

$$= \frac{32\left(9+\sqrt{5}\right) \pm \sqrt{1024\left(86+18\sqrt{5}\right) - 256\left(3-\sqrt{5}\right)\left(33+5\sqrt{5}\right)}}{64\left(3-\sqrt{5}\right)}$$

$$= \frac{32\left(9+\sqrt{5}\right) \pm 16\sqrt{4\left(86+18\sqrt{5}\right) - \left(74-18\sqrt{5}\right)}}{64\left(3-\sqrt{5}\right)} = \frac{2\left(9+\sqrt{5}\right) \pm \sqrt{270+90\sqrt{5}}}{4\left(3-\sqrt{5}\right)}$$

$$= \frac{\left(2\left(9+\sqrt{5}\right) \pm \sqrt{270+90\sqrt{5}}\right)\left(3+\sqrt{5}\right)}{4\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)} = \frac{2\left(9+\sqrt{5}\right)\left(3+\sqrt{5}\right) \pm \left(3+\sqrt{5}\right)\sqrt{9\left(5+\sqrt{5}\right)^2}}{4(4)}$$

$$= \frac{2\left(9+\sqrt{5}\right)\left(3+\sqrt{5}\right) \pm 3\left(3+\sqrt{5}\right)\left(5+\sqrt{5}\right)}{16} = \frac{8\left(8+3\sqrt{5}\right) \pm 12\left(5+2\sqrt{5}\right)}{16}$$

$$= \frac{2\left(8+3\sqrt{5}\right) \pm 3\left(5+2\sqrt{5}\right)}{4} = \frac{\left(16+6\sqrt{5}\right) \pm \left(15+6\sqrt{5}\right)}{16}$$

1. **Taking positive sign, we have**

$$x^2 = \frac{\left(16+6\sqrt{5}\right) + \left(15+6\sqrt{5}\right)}{4} = \frac{31+12\sqrt{5}}{4} \quad or \quad x = \sqrt{\frac{31+12\sqrt{5}}{4}} = \frac{\mathbf{1}}{\mathbf{2}}\sqrt{\mathbf{31+12\sqrt{5}}}$$

Since, $x > 1$ hence, the above value is acceptable.

2. **Taking negative sign, we have**

$$x^2 = \frac{\left(16+6\sqrt{5}\right) - \left(15+6\sqrt{5}\right)}{4} = \frac{1}{4} \quad or \quad x = \sqrt{\frac{1}{4}} = \frac{1}{2} \quad \Rightarrow \boldsymbol{x < 1 \ but \ x > 1} \ (\boldsymbol{required\ condition})$$

Hence, the above value is discarded, now we have

$$x = \frac{1}{2}\sqrt{31+12\sqrt{5}} \ \Rightarrow \frac{R_o}{a} = x = \frac{1}{2}\sqrt{31+12\sqrt{5}} \quad or \quad R_o = \frac{a}{2}\sqrt{31+12\sqrt{5}}$$

Hence, **outer (circumscribed) radius (R_o) of a great rhombicosidodecahedron with edge length a** is given as

$$\boldsymbol{R_o = \frac{a}{2}\sqrt{31+12\sqrt{5}} \approx 3.8023945a} \qquad \ldots\ldots\ldots\ldots\ldots\ldots. \ (VII)$$

Normal distance (H_s) of square faces from the centre of great rhombicosidodecahedron: The normal distance (H_s) of each of 30 congruent square faces from the centre O of a great rhombicosidodecahedron is given from eq(I) as follows

$$H_s = OO_1 = \sqrt{\frac{2{R_o}^2 - a^2}{2}} = \sqrt{\frac{2\left(\frac{a}{2}\sqrt{31+12\sqrt{5}}\right)^2 - a^2}{2}} = a\sqrt{\frac{31+12\sqrt{5}-2}{4}} = \frac{a}{2}\sqrt{29+12\sqrt{5}}$$

$$= \frac{a}{2}\sqrt{\left(3+2\sqrt{5}\right)^2} = \frac{\left(3+2\sqrt{5}\right)a}{2}$$

$$\therefore\ \boldsymbol{H_s = \frac{(3+2\sqrt{5})a}{2} \approx 3.736067978a}$$

It's clear that all 30 congruent square faces are at an equal normal distance H_s from the centre of any great rhombicosidodecahedron.

Solid angle (ω_s) subtended by each of the square faces at the centre of great rhombicosidodecahedron: solid angle $(\boldsymbol{\omega_s})$ subtended by each square face is given from eq(II) as follows

$$\omega_s = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2-a^2}{4x^2-a^2}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2{R_o}^2-a^2}{4{R_o}^2-a^2}}\right) \qquad \left(since,\ \boldsymbol{x = \frac{R_o}{a}}\right)$$

Hence, by substituting the corresponding value of R_o in the above expression, we get

$$\omega_s = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2\left(\frac{a}{2}\sqrt{31+12\sqrt{5}}\right)^2 - a^2}{4\left(\frac{a}{2}\sqrt{31+12\sqrt{5}}\right)^2 - a^2}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{31+12\sqrt{5}-2}{2(31+12\sqrt{5}-1)}}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\sqrt{\frac{29+12\sqrt{5}}{12(5+2\sqrt{5})}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{(29+12\sqrt{5})(5-2\sqrt{5})}{12(5+2\sqrt{5})(5-2\sqrt{5})}}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\sqrt{\frac{25+2\sqrt{5}}{60}}\right)$$

$$\therefore\ \boldsymbol{\omega_s = 2\pi - 8\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{25+2\sqrt{5}}{15}}\right) = 4\sin^{-1}\left(\frac{5-2\sqrt{5}}{30}\right) \approx 0.070385504\ sr}$$

Normal distance $(\boldsymbol{H_h})$ of regular hexagonal faces from the centre of great rhombicosidodecahedron: The normal distance (H_h) of each of 20 congruent regular hexagonal faces from the centre O of a great rhombicosidodecahedron is given from eq(III) as follows

$$H_h = OO_2 = \sqrt{{R_o}^2 - a^2} = \sqrt{\left(\frac{a}{2}\sqrt{31+12\sqrt{5}}\right)^2 - a^2} = a\sqrt{\frac{31+12\sqrt{5}-4}{4}} = \frac{a}{2}\sqrt{27+12\sqrt{5}}$$

$$= \frac{a}{2}\sqrt{3(2+\sqrt{5})^2} = \frac{\sqrt{3}(2+\sqrt{5})a}{2}$$

$$\therefore\ \boldsymbol{H_h = \frac{\sqrt{3}(2+\sqrt{5})a}{2} \approx 3.668542481a}$$

It's clear that all 20 congruent regular hexagonal faces are at an equal normal distance H_h from the centre of any great rhombicosidodecahedron.

Solid angle (ω_h) subtended by each of the regular hexagonal faces at the centre of great rhombicosidodecahedron: solid angle $(\boldsymbol{\omega_h})$ subtended by each regular hexagonal face is given from eq(IV) as follows

$$\omega_h = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{x^2-a^2}{4x^2-a^2}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{{R_o}^2-a^2}{4{R_o}^2-a^2}}\right) \qquad \left(since,\ \boldsymbol{x = \frac{R_o}{a}}\right)$$

Hence, by substituting the corresponding value of R_o in the above expression, we get

$$\omega_h = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{\left(\frac{a}{2}\sqrt{31+12\sqrt{5}}\right)^2 - a^2}{4\left(\frac{a}{2}\sqrt{31+12\sqrt{5}}\right)^2 - a^2}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{31+12\sqrt{5}-4}{4(31+12\sqrt{5}-1)}}\right)$$

$$= 2\pi - 12\sin^{-1}\left(\sqrt{\frac{9+4\sqrt{5}}{8(5+2\sqrt{5})}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{(9+4\sqrt{2})(5-2\sqrt{5})}{8(5+2\sqrt{5})(5-2\sqrt{5})}}\right)$$

$$= 2\pi - 12\sin^{-1}\left(\sqrt{\frac{5+2\sqrt{5}}{40}}\right) = 2\pi - 12\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{5+2\sqrt{5}}{10}}\right)$$

$$\therefore \ \boldsymbol{\omega_h = 2\pi - 12\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{5+2\sqrt{5}}{10}}\right) \approx 0.184524626\ sr}$$

Normal distance (H_D) of regular decagonal faces from the centre of great rhombicosidodecahedron: The normal distance (H_D) of each of 12 congruent regular decagonal faces from the centre O of a great rhombicosidodecahedron is given from eq(V) as follows

$$H_D = OO_3 = \sqrt{\frac{2R_o^2 - (3+\sqrt{5})a^2}{2}} = \sqrt{\frac{2\left(\frac{a}{2}\sqrt{31+12\sqrt{5}}\right)^2 - (3+\sqrt{5})a^2}{2}} = \frac{a}{2}\sqrt{31+12\sqrt{5}-2(3+\sqrt{5})}$$

$$= \frac{a}{2}\sqrt{25+10\sqrt{5}}$$

$$\therefore \ \boldsymbol{H_D = \frac{a}{2}\sqrt{25+10\sqrt{5}} \approx 3.440954801a}$$

It's clear that all 12 congruent regular decagonal faces are at an equal normal distance H_D from the centre of any great rhombicosidodecahedron.

Solid angle (ω_D) subtended by each of the regular decagonal faces at the centre of great rhombicosidodecahedron: solid angle $(\boldsymbol{\omega_D})$ subtended by each regular decagonal face is given from eq(VI) as follows

$$\omega_D = 2\pi - 20\sin^{-1}\left(\sqrt{\frac{\left(\frac{3-\sqrt{5}}{2}\right)x^2 - 1}{4x^2 - 1}}\right) = 2\pi - 20\sin^{-1}\left(\sqrt{\frac{\left(\frac{3-\sqrt{5}}{2}\right)R_o^2 - a^2}{4R_o^2 - a^2}}\right) \quad \left(since,\ \boldsymbol{x = \frac{R_o}{a}}\right)$$

Hence, by substituting the corresponding value of R_o in the above expression, we get

$$\omega_D = 2\pi - 20\sin^{-1}\left(\sqrt{\frac{\left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{a}{2}\sqrt{31+12\sqrt{5}}\right)^2 - a^2}{4\left(\frac{a}{2}\sqrt{31+12\sqrt{5}}\right)^2 - a^2}}\right)$$

$$= 2\pi - 20\sin^{-1}\left(\sqrt{\frac{(3-\sqrt{5})(31+12\sqrt{5}) - 8}{8(31+12\sqrt{5}-1)}}\right)$$

$$= 2\pi - 20\sin^{-1}\left(\sqrt{\frac{25+5\sqrt{5}}{48(5+2\sqrt{5})}}\right) = 2\pi - 20\sin^{-1}\left(\sqrt{\frac{(25+5\sqrt{5})(5-2\sqrt{5})}{48(5+2\sqrt{5})(5-2\sqrt{5})}}\right)$$

$$= 2\pi - 20\sin^{-1}\left(\sqrt{\frac{75-25\sqrt{5}}{48(5)}}\right) = 2\pi - 20\sin^{-1}\left(\sqrt{\frac{15-5\sqrt{5}}{48}}\right) = 2\pi - 20\sin^{-1}\left(\sqrt{\frac{30-10\sqrt{5}}{96}}\right)$$

$$= 2\pi - 20\sin^{-1}\left(\sqrt{\frac{(5-\sqrt{5})^2}{96}}\right) = 2\pi - 20\sin^{-1}\left(\frac{5-\sqrt{5}}{4\sqrt{6}}\right)$$

$$\therefore \quad \boldsymbol{\omega_D = 2\pi - 20\sin^{-1}\left(\frac{5-\sqrt{5}}{4\sqrt{6}}\right) \approx 0.563692745\ sr}$$

It's clear from the above results that the solid angle subtended by each of 12 regular decagonal faces is greater than the solid angle subtended by each of 30 square faces & each of 20 regular hexagonal faces at the centre of any great rhombicosidodecahedron.

It's also clear from the above results that $\boldsymbol{H_s > H_h > H_D}$ i.e. the normal distance (H_s) of square faces is greater than the normal distance H_h of the regular hexagonal faces & the normal distance H_D of the regular decagonal faces from the centre of a great rhombicosidodecahedron i.e. **regular decagonal faces are closer to the centre as compared to the square & regular hexagonal faces in any great rhombicosidodecahedron**.

Important parameters of a great rhombicosidodecahedron:

1. **Inner (inscribed) radius ($\boldsymbol{R_i}$):** It is the radius of the largest sphere inscribed (trapped inside) by a great rhombicosidodecahedron. The largest inscribed sphere always touches all 12 congruent regular decagonal faces but does not touch any of 30 congruent square & any of 20 congruent regular hexagonal faces at all since all 12 decagonal faces are closest to the centre in all the faces. Thus, the inner radius is always equal to the normal distance (H_D) of regular decagonal faces from the centre of a great rhombicosidodecahedron & is given as follows

$$\boldsymbol{R_i = H_D = \frac{a}{2}\sqrt{25+10\sqrt{5}} \approx 3.440954801a}$$

Hence, the **volume of inscribed sphere** is given as

$$V_{inscribed} = \frac{4}{3}\pi(R_i)^3 = \boldsymbol{\frac{4}{3}\pi\left(\frac{a}{2}\sqrt{25+10\sqrt{5}}\right)^3 \approx 170.6575526a^3}$$

2. **Outer (circumscribed) radius ($\boldsymbol{R_o}$):** It is the radius of the smallest sphere circumscribing a great rhombicosidodecahedron or it's the radius of a spherical surface passing through all 120 vertices of a great rhombicosidodecahedron. It is from the eq(VII) as follows

$$\boldsymbol{R_o = \frac{a}{2}\sqrt{31+12\sqrt{5}} \approx 3.8023945a}$$

Hence, the **volume of circumscribed sphere** is given as

$$V_{circumscribed} = \frac{4}{3}\pi(R_o)^3 = \boldsymbol{\frac{4}{3}\pi\left(\frac{a}{2}\sqrt{31+12\sqrt{5}}\right)^3 = 230.2820721a^3}$$

3. **Surface area (A_s):** We know that a great rhombicosidodecahedron has 30 congruent square faces, 20 congruent regular hexagonal faces & 12 congruent regular decagonal faces each of edge length a. Hence, its surface area is given as follows

$$A_s = 30(area\ of\ square) + 20(area\ of\ regular\ hexagon) + 12(area\ of\ regular\ decagon)$$

We know that **area of any regular n-polygon** with each side of length a is given as

$$\boldsymbol{A = \frac{1}{4}na^2\cot\frac{\pi}{n}}$$

Hence, by substituting all the corresponding values in the above expression, we get

$$A_s = 30 \times \left(\frac{1}{4} \times 4a^2\cot\frac{\pi}{4}\right) + 20 \times \left(\frac{1}{4} \times 6a^2\cot\frac{\pi}{6}\right) + 12 \times \left(\frac{1}{4} \times 10a^2\cot\frac{\pi}{10}\right)$$

$$= 30a^2 + 30\sqrt{3}a^2 + 30a^2\sqrt{5+2\sqrt{5}} = 30\left(1+\sqrt{3}+\sqrt{5+2\sqrt{5}}\right)a^2$$

$$\boldsymbol{A_s = 30\left(1+\sqrt{3}+\sqrt{5+2\sqrt{5}}\right)a^2 \approx 174.2920303a^2}$$

4. **Volume (V):** We know that a great rhombicosidodecahedron with edge length a has 30 congruent square faces, 20 congruent regular hexagonal faces & 12 congruent regular decagonal faces. Hence, the volume (V) of the great rhombicosidodecahedron is the sum of volumes of all its elementary right pyramids with square base, regular hexagonal base & regular decagonal base (face) (see figure 2 above) & is given as follows

$$V = 30(volume\ of\ right\ pyramid\ with\ square\ base) + 20(volume\ of\ right\ pyramid\ with\ regular\ hexagonal\ base) + 12(volume\ of\ right\ pyramid\ with\ regular\ decagonal\ base)$$

$$= 30\left(\frac{1}{3}(area\ of\ square) \times H_s\right) + 20\left(\frac{1}{3}(area\ of\ regular\ hexagon) \times H_h\right) + 12\left(\frac{1}{3}(area\ of\ regular\ decagon) \times H_D\right)$$

$$= 30\left(\frac{1}{3}\left(\frac{1}{4} \times 4a^2\cot\frac{\pi}{4}\right) \times \frac{(3+2\sqrt{5})a}{2}\right) + 20\left(\frac{1}{3}\left(\frac{1}{4} \times 6a^2\cot\frac{\pi}{6}\right) \times \frac{\sqrt{3}(2+\sqrt{5})a}{2}\right) + 12\left(\frac{1}{3}\left(\frac{1}{4} \times 10a^2\cot\frac{\pi}{10}\right) \times \frac{a}{2}\sqrt{25+10\sqrt{5}}\right)$$

$$= 5(3+2\sqrt{5})a^3 + 15(2+\sqrt{5})a^3 + 5a^3\sqrt{(5+2\sqrt{5})}\sqrt{(25+10\sqrt{5})}$$

$$= 5(9+5\sqrt{5})a^3 + 5a^3\sqrt{(10+5\sqrt{5})^2} = 5(9+5\sqrt{5})a^3 + 5(10+5\sqrt{5})a^3 = (95+50\sqrt{5})a^3$$

$$\boldsymbol{V = (95+50\sqrt{5})a^3 \approx 206.8033989a^3}$$

5. **Mean radius (R_m):** It is the radius of the sphere having a volume equal to that of a great rhombicosidodecahedron. It is calculated as follows

$$volume\ of\ sphere\ with\ mean\ radius\ R_m = volume\ of\ the\ great\ rhombicosidodecahedron$$

$$\frac{4}{3}\pi(R_m)^3 = \left(95+50\sqrt{5}\right)a^3 \quad \Rightarrow (R_m)^3 = \frac{3\left(95+50\sqrt{5}\right)a^3}{4\pi} \quad or\ R_m = a\left(\frac{15\left(19+10\sqrt{2}\right)}{4\pi}\right)^{\frac{1}{3}}$$

$$\boldsymbol{R_m = a\left(\frac{15\left(19+10\sqrt{5}\right)}{4\pi}\right)^{\frac{1}{3}} \approx 3.668509834a}$$

It's clear from above results that $\boldsymbol{R_i < R_m < R_o}$

6. **Dihedral angles between the adjacent faces:** In order to calculate dihedral angles between the different adjacent faces with a common edge in a great rhombicosidodecahedron, let's consider one-by-one all three pairs of adjacent faces with a common edge as follows

a. **Angle between square face & regular hexagonal face:** Draw the perpendiculars OO_1 & OO_2 from the centre O of great rhombicosidodecahedron to the square face & the regular hexagonal face which have a common edge (See figure 3). We know that the inscribed radius (r_i) of any regular n-gon with each side a is given as follows

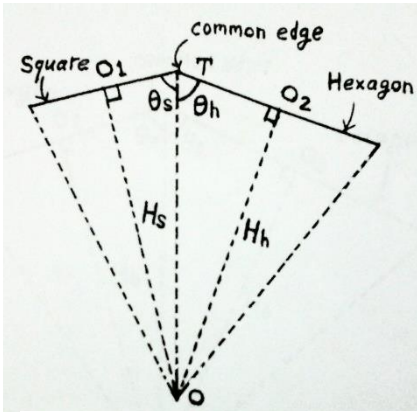

Figure 3: Square face with centre $\boldsymbol{O_1}$ & regular hexagonal face with centre $\boldsymbol{O_2}$ with a common edge (denoted by point T) normal to the plane of paper

$$\boldsymbol{r_i = inscribed\ radius\ of\ any\ regular\ n-gon\ = \frac{a}{2}cot\frac{\pi}{n}}$$

$$\therefore\ O_1T = inscribed\ radius\ of\ square = \frac{a}{2}cot\frac{\pi}{4} = \frac{a}{2}\ \ \&$$

$$\therefore\ O_2T = inscribed\ radius\ of\ regular\ hexagon = \frac{a}{2}cot\frac{\pi}{6} = \frac{a\sqrt{3}}{2}$$

In right ΔOO_1T

$$tan\theta_s = \frac{OO_1}{O_1T} = \frac{H_s}{\left(\frac{a}{2}\right)} = \frac{\left(\frac{\left(3+2\sqrt{5}\right)a}{2}\right)}{\left(\frac{a}{2}\right)} = \left(3+2\sqrt{5}\right)$$

$$\therefore\ \boldsymbol{\theta_s = \tan^{-1}\left(3+2\sqrt{5}\right) \approx 82.37736814^o} \qquad \dots\dots\dots\dots\dots\dots(VIII)$$

In right ΔOO_2T

$$tan\theta_h = \frac{OO_2}{O_2T} = \frac{H_h}{\left(\frac{a\sqrt{3}}{2}\right)} = \frac{\left(\frac{\sqrt{3}\left(2+\sqrt{5}\right)a}{2}\right)}{\left(\frac{a\sqrt{3}}{2}\right)} = \left(2+\sqrt{5}\right)$$

$$\therefore\ \boldsymbol{\theta_h = \tan^{-1}\left(2+\sqrt{5}\right) \approx 76.71747441^o} \qquad \dots\dots\dots\dots\dots\dots(IX)$$

$$\Rightarrow \theta_s + \theta_h = \tan^{-1}\left(3+2\sqrt{5}\right) + \tan^{-1}\left(2+\sqrt{5}\right) = \tan^{-1}\left(\frac{\left(3+2\sqrt{5}\right)+\left(2+\sqrt{5}\right)}{1-\left(3+2\sqrt{5}\right)\left(2+\sqrt{5}\right)}\right)$$

$$= \tan^{-1}\left(\frac{5+3\sqrt{5}}{1-\left(16+7\sqrt{5}\right)}\right) = \tan^{-1}\left(\frac{-\left(5+3\sqrt{5}\right)}{15+7\sqrt{5}}\right) = \pi - \tan^{-1}\left(\frac{5+3\sqrt{5}}{15+7\sqrt{5}}\right)$$

$$= \pi - \tan^{-1}\left(\frac{(5+3\sqrt{5})(7\sqrt{5}-15)}{(7\sqrt{5}+15)(7\sqrt{5}-15)}\right) = \pi - \tan^{-1}\left(\frac{30-10\sqrt{5}}{20}\right) = \pi - \tan^{-1}\left(\frac{3-\sqrt{5}}{2}\right)$$

Hence**, dihedral angle between the square face & the regular hexagonal face is given as**

$$\boldsymbol{\theta_s + \theta_h = \pi - \tan^{-1}\left(\frac{3-\sqrt{5}}{2}\right) \approx 159.0948426^o}$$

b. Angle between square face & regular decagonal face: Draw the perpendiculars OO_1 & OO_3 from the centre O of great rhombicosidodecahedron to the square face & the regular decagonal face which have a common edge (See figure 4).

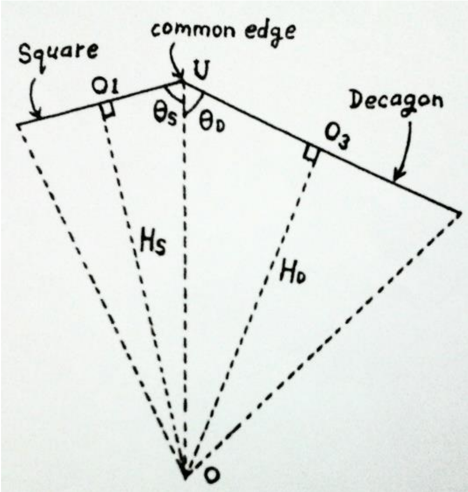

Figure 4: Square face with centre O_1 & regular decagonal face with centre O_3 with a common edge (denoted by point U) normal to the plane of paper

$$O_3U = inscribed\ radius\ of\ regular\ decagon = \frac{a}{2}cot\frac{\pi}{10}$$

$$= \frac{a\sqrt{5+2\sqrt{5}}}{2}$$

In right ΔOO_3U

$$tan\theta_D = \frac{OO_3}{O_3U} = \frac{H_D}{\left(\frac{a\sqrt{5+2\sqrt{5}}}{2}\right)} = \frac{\left(\frac{a}{2}\sqrt{25+10\sqrt{5}}\right)}{\left(\frac{a\sqrt{5+2\sqrt{5}}}{2}\right)} = \sqrt{\frac{5(5+2\sqrt{5})}{5+2\sqrt{5}}} = \sqrt{5}$$

$$\therefore\ \boldsymbol{\theta_D = \tan^{-1}(\sqrt{5}) \approx 65.90515745^o} \qquad \dots\dots\dots\dots\dots\dots (X)$$

$$\Rightarrow \theta_s + \theta_D = \tan^{-1}(3+2\sqrt{5}) + \tan^{-1}(\sqrt{5}) = \tan^{-1}\left(\frac{(3+2\sqrt{5})+(\sqrt{5})}{1-(3+2\sqrt{5})(\sqrt{5})}\right)$$

$$= \tan^{-1}\left(\frac{(3+3\sqrt{5})}{1-(10+3\sqrt{5})}\right) = \tan^{-1}\left(\frac{-3(1+\sqrt{5})}{9+3\sqrt{5}}\right) = \pi - \tan^{-1}\left(\frac{1+\sqrt{5}}{3+\sqrt{5}}\right)$$

$$= \pi - \tan^{-1}\left(\frac{(1+\sqrt{5})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})}\right) = \pi - \tan^{-1}\left(\frac{2(\sqrt{5}-1)}{4}\right) = \pi - \tan^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$$

Hence**, dihedral angle between the square face & the regular decagonal face is given as**

$$\boldsymbol{\theta_s + \theta_D = \pi - \tan^{-1}\left(\frac{\sqrt{5}-1}{2}\right) \approx 148.2825256^o}$$

c. Angle between regular hexagonal face & regular decagonal face: Draw the perpendiculars OO_2 & OO_3 from the centre O of great rhombicosidodecahedron to the regular hexagonal face & the regular decagonal face which have a common edge (See figure 5). Now from eq(IX) & (X), we get

$$\theta_h + \theta_D = \tan^{-1}(2+\sqrt{5}) + \tan^{-1}(\sqrt{5}) = \tan^{-1}\left(\frac{(2+\sqrt{5})+(\sqrt{5})}{1-(2+\sqrt{5})(\sqrt{5})}\right) = \tan^{-1}\left(\frac{(2+2\sqrt{5})}{1-(5+2\sqrt{5})}\right)$$

$$= \tan^{-1}\left(\frac{-2(1+\sqrt{5})}{4+2\sqrt{5}}\right) = \pi - \tan^{-1}\left(\frac{1+\sqrt{5}}{2+\sqrt{5}}\right)$$

$$= \pi - \tan^{-1}\left(\frac{(1+\sqrt{5})(\sqrt{5}-2)}{(\sqrt{5}+2)(\sqrt{5}-2)}\right) = \pi - \tan^{-1}\left(\frac{3-\sqrt{5}}{1}\right) = \pi - \tan^{-1}(3-\sqrt{5})$$

Hence, **dihedral angle between the regular hexagonal face & the regular decagonal face is given as**

$$\boldsymbol{\theta_h + \theta_D = \pi - \tan^{-1}(3-\sqrt{5}) \approx 142.6226319^o}$$

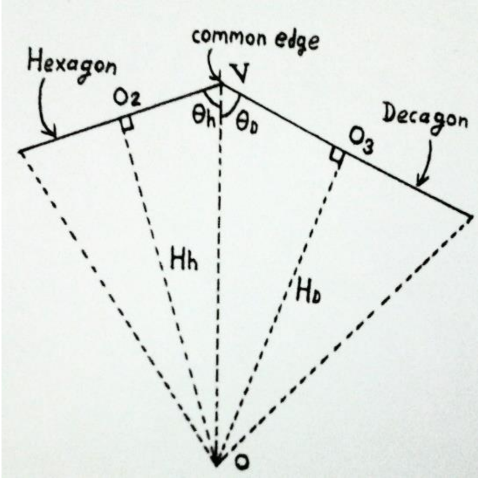

Figure 5: Regular hexagonal face with centre O_2 & regular decagonal face with centre O_3 with a common edge (denoted by point V) normal to the plane of paper

Construction of a solid great rhombicosidodecahedron: In order to construct a solid great rhombicosidodecahedron with edge length a there are two methods

1. Construction from elementary right pyramids: In this method, first we construct all elementary right pyramids as follows

Construct 30 congruent right pyramids with square base of side length a & normal height (H_s)

$$H_s = \frac{(3+2\sqrt{5})a}{2} \approx 3.736067978a$$

Construct 20 congruent right pyramids with regular hexagonal base of side length a & normal height (H_h)

$$H_h = \frac{\sqrt{3}(2+\sqrt{5})a}{2} \approx 3.668542481a$$

Construct 12 congruent right pyramids with regular decagonal base of side length a & normal height (H_D)

$$H_D = \frac{a}{2}\sqrt{25+10\sqrt{5}} \approx 3.440954801a$$

Now, paste/bond by joining all these elementary right pyramids by overlapping their lateral surfaces & keeping their apex points coincident with each other such that 4 edges of each square base (face) coincide with the edges of 2 regular hexagonal bases & 2 regular decagonal bases (faces). Thus a solid great rhombicosidodecahedron, with 30 congruent square faces, 20 congruent regular hexagonal faces & 12 congruent regular decagonal faces each of edge length a, is obtained.

2. Facing a solid sphere: It is a method of facing, first we select a **blank as a solid sphere** of certain material (i.e. metal, alloy, composite material etc.) & with suitable diameter in order to obtain the maximum desired edge length of a great rhombicosidodecahedron. Then, we perform the facing operations on the solid sphere to generate 30 congruent square faces, 20 congruent regular hexagonal faces & 12 congruent regular decagonal faces each of equal edge length.

Let there be a blank as a solid sphere with a diameter D. Then the edge length a, of a great rhombicosidodecahedron of the maximum volume to be produced, can be co-related with the diameter D by **relation of outer radius (R_o) with edge length (a) of the great rhombicosidodecahedron** as follows

$$R_o = \frac{a}{2}\sqrt{31+12\sqrt{5}}$$

Now, substituting $R_o = {}^{D}/_{2}$ in the above expression, we have

$$\frac{D}{2} = \frac{a}{2}\sqrt{31 + 12\sqrt{5}} \quad or \quad a = \frac{D}{\sqrt{31 + 12\sqrt{5}}}$$

$$\boldsymbol{a = \frac{D}{\sqrt{31 + 12\sqrt{5}}} \approx 0.131496087D}$$

Above relation is very useful for determining the edge length a of a great rhombicosidodecahedron to be produced from a solid sphere with known diameter D for manufacturing purpose.

Hence, the **maximum volume of great rhombicosidodecahedron** produced from a solid sphere is given as follows

$$V_{max} = (95 + 50\sqrt{5})a^3 = (95 + 50\sqrt{5})\left(\frac{D}{\sqrt{31 + 12\sqrt{5}}}\right)^3 = \frac{(95 + 50\sqrt{5})D^3}{(31 + 12\sqrt{5})\sqrt{31 + 12\sqrt{5}}}$$

$$= \frac{(95 + 50\sqrt{5})(31 - 12\sqrt{5})D^3}{241\sqrt{31 + 12\sqrt{5}}} = \frac{(-55 + 410\sqrt{5})D^3}{241\sqrt{31 + 12\sqrt{5}}} = \frac{5(82\sqrt{5} - 11)D^3}{241\sqrt{31 + 12\sqrt{5}}}$$

$$\boldsymbol{V_{max} = \frac{5(82\sqrt{5} - 11)D^3}{241\sqrt{31 + 12\sqrt{5}}} \approx 0.470214661D^3}$$

M**inimum volume of material removed** is given as

$$(V_{removed})_{min} = (volume\ of\ parent\ sphere\ with\ diameter\ D) - (volume\ of\ great\ rhombicosidodecahedron)$$

$$= \frac{\pi}{6}D^3 - \frac{5(82\sqrt{5} - 11)D^3}{241\sqrt{31 + 12\sqrt{5}}} = \left(\frac{\pi}{6} - \frac{5(82\sqrt{5} - 11)}{241\sqrt{31 + 12\sqrt{5}}}\right)D^3$$

$$\boldsymbol{(V_{removed})_{min} = \left(\frac{\pi}{6} - \frac{5(82\sqrt{5} - 11)}{241\sqrt{31 + 12\sqrt{5}}}\right)D^3 \approx 0.053384114D^3}$$

Percentage $(\%)$ of minimum volume of material removed

$$\%\ \boldsymbol{of}\ \boldsymbol{V_{removed}} = \frac{minimum\ volume\ removed}{total\ volume\ of\ sphere} \times 100$$

$$= \frac{\left(\frac{\pi}{6} - \frac{5(82\sqrt{5} - 11)}{241\sqrt{31 + 12\sqrt{5}}}\right)D^3}{\frac{\pi}{6}D^3} \times 100 = \boldsymbol{\left(1 - \frac{30(82\sqrt{5} - 11)}{241\pi\sqrt{31 + 12\sqrt{5}}}\right) \times 100 \approx 10.19\%}$$

It's obvious that when a great rhombicosidodecahedron of the maximum volume is produced from a solid sphere then about $\boldsymbol{10.19\%}$ volume of material is removed as scraps. Thus, we can select optimum diameter of blank as a solid sphere to produce a solid great rhombicosidodecahedron of the maximum volume (or with the maximum desired edge length)

Conclusions: Let there be any great rhombicosidodecahedron having 30 congruent square faces, 20 congruent regular hexagonal faces & 12 congruent regular decagonal faces each with edge length a then all its important parameters are calculated/determined as tabulated below

Congruent polygonal faces	**No. of faces**	**Normal distance of each face from the centre of the great rhombicosidodecahedron**	**Solid angle subtended by each face at the centre of the great rhombicosidodecahedron**
Square	30	$\frac{(3+2\sqrt{5})a}{2} \approx 3.736067978a$	$4\sin^{-1}\left(\frac{5-2\sqrt{5}}{30}\right) \approx 0.070385504\ sr$
Regular hexagon	20	$\frac{\sqrt{3}(2+\sqrt{5})a}{2} \approx 3.668542481a$	$2\pi - 12\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{5+2\sqrt{5}}{10}}\right) \approx 0.184524626\ sr$
Regular decagon	12	$\frac{a}{2}\sqrt{25+10\sqrt{5}} \approx 3.440954801a$	$2\pi - 20\sin^{-1}\left(\frac{5-\sqrt{5}}{4\sqrt{6}}\right) \approx 0.563692745\ sr$

Inner (inscribed) radius (R_i)	$R_i = \frac{a}{2}\sqrt{25+10\sqrt{5}} \approx 3.440954801a$
Outer (circumscribed) radius (R_o)	$R_o = \frac{a}{2}\sqrt{31+12\sqrt{5}} \approx 3.8023945a$
Mean radius (R_m)	$R_m = a\left(\frac{15(19+10\sqrt{5})}{4\pi}\right)^{\frac{1}{3}} \approx 3.668509834a$
Surface area (A_s)	$A_s = 30\left(1+\sqrt{3}+\sqrt{5+2\sqrt{5}}\right)a^2 \approx 174.2920303a^2$
Volume (V)	$V = (95+50\sqrt{5})a^3 \approx 206.8033989a^3$

Table for the dihedral angles between the adjacent faces of a great rhombicosidodecahedron

Pair of the adjacent faces with a common edge	Square & regular hexagon	Square & regular decagon	Regular hexagon & regular decagon
Dihedral angle of the corresponding pair (of the adjacent faces)	$\theta_s + \theta_h = \pi - \tan^{-1}\left(\frac{3-\sqrt{5}}{2}\right)$ $\approx 159.0948426^o$	$\theta_s + \theta_D = \pi - \tan^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$ $\approx 148.2825256^o$	$\theta_h + \theta_D = \pi - \tan^{-1}(3-\sqrt{5})$ $\approx 142.6226319^o$

Note: Above articles had been developed & *illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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