Solid angle subtended by a beam with rectangular profile emitted by uniformly radiating point-source given horizontal & vertical beam angles

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Introduction: Here, we are interested to find out the general expressions for calculating the solid angle subtended by a beam, having a **rectangular profile**, emitted by a **uniformly radiating point-source** given **horizontal & vertical beam angles $\boldsymbol{\theta_H}$ & $\boldsymbol{\theta_V}$** respectively measured in two centralised orthogonal (horizontal & vertical) planes passing through the point-source, as well as for the given values of **lateral beam angles $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$** measured between the adjacent lateral beam rays (passing through the vertices of rectangular profile) originated from the point-source (As shown in the figure 1) by using the **standard formula of solid angle of a rectangular plane**. This standard formula had already been derived by the author in his paper "**HCR's Theory of Polygon**". Subsequently, both the cases will be discussed & analysed in an order.

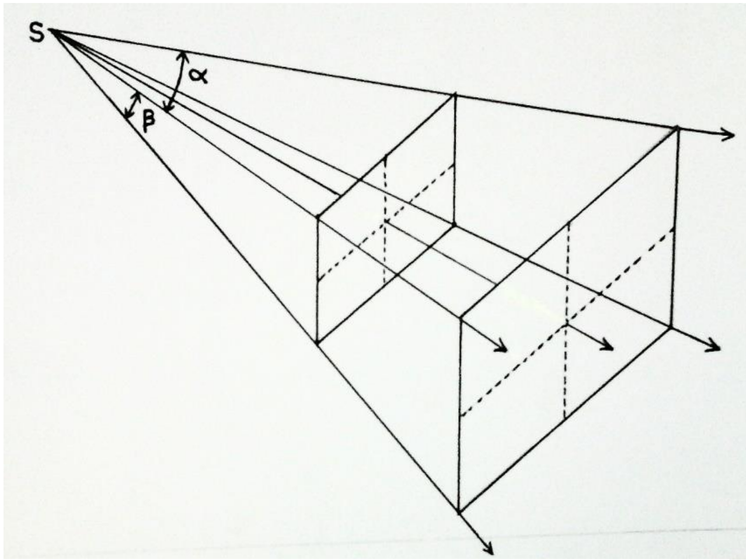

Figure 1: A beam with rectangular profile is emitted by a uniform point-source S. The angles $\boldsymbol{\theta_H}$ & $\boldsymbol{\theta_V}$ are the horizontal & vertical beam angles (i.e. angles subtended by the centralised dotted lines at the point-source S) and $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$ are the lateral beam angles (i.e. angles between the lateral beam rays)

1. Solid angle subtended by the beam with rectangular profile given horizontal & vertical beam angles $\boldsymbol{\theta_H}$ & $\boldsymbol{\theta_V}$:

Let there be a beam with rectangular profile radiated from a uniform point-source S such that θ_H & θ_V are (plane) angles subtended by the beam in the centralised horizontal & vertical planes respectively passing through the point-source S. Now, consider an imaginary rectangular plane ABCD with centre O, length l & width b at a normal distance $OS = h$ from the uniform point-source S (As shown in the figure 2)

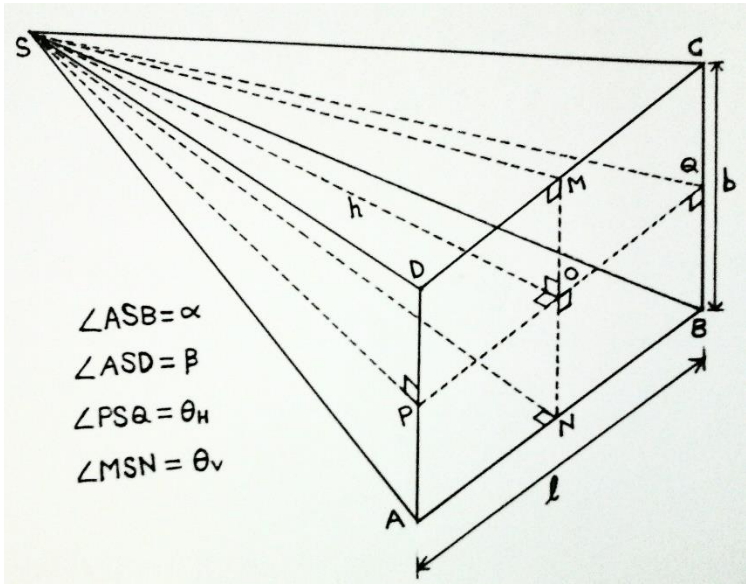

Figure 2: An imaginary rectangular plane ABCD, having centre O, length $\boldsymbol{l}$ & width $\boldsymbol{b}$ lying at a normal distance $\boldsymbol{OS = h}$ from the uniform point-source S, is representing the rectangular profile of the beam.

In right ΔSOP

$$\tan\angle OSP = \frac{OP}{OS} \quad \Rightarrow \tan\frac{\angle PSQ}{2} = \frac{OP}{OS}$$

$$\Rightarrow \tan\frac{\theta_H}{2} = \frac{\frac{l}{2}}{h} \quad \Rightarrow \boldsymbol{l = 2h\tan\frac{\theta_H}{2}}$$

Similarly, in right ΔSOM

$$\tan\angle OSM = \frac{OM}{OS} \quad \Rightarrow \tan\frac{\angle MSN}{2} = \frac{OM}{OS}$$

$$\Rightarrow \tan\frac{\theta_V}{2} = \frac{\frac{b}{2}}{h} \quad \Rightarrow \boldsymbol{b = 2h\tan\frac{\theta_V}{2}}$$

Now, the solid angle (ω) subtended by the rectangular plane ABCD, having length l & width b lying at a normal distance $OS = h$, at the point-source S is given by the **standard formula of rectangular plane** as follows

$$\boldsymbol{\omega = 4\sin^{-1}\left(\frac{lb}{\sqrt{(l^2+4h^2)(b^2+4h^2)}}\right)} \qquad (as\ derived\ in\ \boldsymbol{HCR's\ Theory\ of\ Polygon})$$

Now, substituting the values of length l & width b in the above expression, we get

$$\omega = 4\sin^{-1}\left(\frac{\left(2h\tan\frac{\theta_H}{2}\right)\left(2h\tan\frac{\theta_V}{2}\right)}{\sqrt{\left(\left(2h\tan\frac{\theta_H}{2}\right)^2+4h^2\right)\left(\left(2h\tan\frac{\theta_V}{2}\right)^2+4h^2\right)}}\right)$$

$$= 4\sin^{-1}\left(\frac{4h^2\tan\frac{\theta_H}{2}\tan\frac{\theta_V}{2}}{4h^2\sqrt{\left(1+\tan^2\frac{\theta_H}{2}\right)\left(\left(1+\tan^2\frac{\theta_V}{2}\right)\right)}}\right) = 4\sin^{-1}\left(\frac{\tan\frac{\theta_H}{2}\tan\frac{\theta_V}{2}}{\sec\frac{\theta_H}{2}\sec\frac{\theta_V}{2}}\right) = 4\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)$$

Hence, the **solid angle ($\boldsymbol{\omega}$) subtended by a beam with rectangular profile at the uniform point-source given the horizontal & vertical (plane) beam angles $\boldsymbol{\theta_H}$ & $\boldsymbol{\theta_V}$**, is given as

$$\boldsymbol{\omega = 4\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)} \qquad (\forall\quad \boldsymbol{\theta_H, \theta_V \in [0,\pi]})$$

Note: If $\boldsymbol{\theta_H, \theta_V \geq \pi}$ then apply the following formula to get solid angle subtended by the beam

$$\boldsymbol{\omega = 4\pi - 4\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)} \qquad (\forall\quad \boldsymbol{\theta_H, \theta_V \in [\pi, 2\pi]})$$

Surface area intercepted by the beam with a spherical surface: The surface area intercepted by the beam, having rectangular profile with horizontal & vertical beam angles θ_H & θ_V, with the spherical surface having a radius R & centre at the point-source is given as

$$A_S = (solid\ angle) \times (radius\ of\ spherical\ surface)^2 = \omega R^2 = 4R^2\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)$$

$$\boldsymbol{Spherical\ surface\ area\ intercepted\ by\ the\ beam,\ \ A_S = 4R^2\sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)}$$

Where, $\boldsymbol{\theta_H, \theta_V \in [0, \pi]} \quad or \quad \boldsymbol{0 \leq \theta_H, \theta_V \leq \pi}$

$$\boldsymbol{Spherical\ surface\ area\ intercepted\ by\ the\ beam,\ \ A_S = 4R^2\left(\pi - \sin^{-1}\left(\sin\frac{\theta_H}{2}\sin\frac{\theta_V}{2}\right)\right)}$$

Where, $\boldsymbol{\theta_H, \theta_V \in [\pi,\ 2\pi]} \quad or \quad \boldsymbol{\pi \leq \theta_H, \theta_V \leq 2\pi}$

Beam with a circular profile equivalent to the beam with a rectangular profile emitted from the same uniform point-source: The beams with the circular & the rectangular profiles are said to be equivalent to each other if they subtend an equal solid angle at a given uniform point-source. Thus profile of the original beam is changed without any change in the total energy\luminous flux associated with the original beam. Now, let's consider a beam with circular profile subtending a cone angle θ_C & a solid angle ω at the uniform point-source S (As shown in the figure 3)

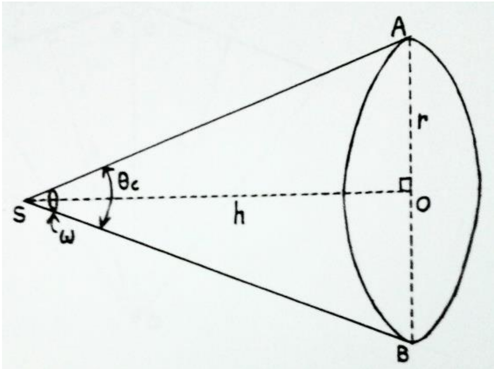

Figure 3: An imaginary circular plane, having centre O & a radius r lying at a normal distance $OS = h$ from the uniform point-source S, is representing the circular profile of the beam with a cone angle θ_C

Now, consider an imaginary circular plane, with centre O & a radius r, at a normal distance h from the point source S.

In right ΔAOS

$$cos\angle ASO = \frac{OS}{AS} \Rightarrow cos\frac{\angle ASB}{2} = \frac{OS}{\sqrt{(OS)^2 + (OA)^2}}$$

$$\Rightarrow \frac{\boldsymbol{h}}{\sqrt{\boldsymbol{h}^2 + \boldsymbol{r}^2}} = \boldsymbol{cos}\frac{\boldsymbol{\theta_C}}{\boldsymbol{2}} \qquad \dots\dots\dots\dots\dots\dots..(1)$$

Now, the solid angle (ω) subtended by the circular plane at the point-source S is by the **standard formula of solid angle of circular plane** as follows

$$\omega = 2\pi\left(1 - \frac{h}{\sqrt{h^2 + r^2}}\right) = 2\pi\left(1 - cos\frac{\theta_C}{2}\right) \quad (by\ setting\ the\ value\ from\ eq(1))$$

Now, equating the values of solid angles subtended by the beams with circular & rectangular profiles at the given uniform point source S, as follows

$$2\pi\left(1 - cos\frac{\theta_C}{2}\right) = 4\sin^{-1}\left(sin\frac{\theta_H}{2}sin\frac{\theta_V}{2}\right)$$

$$\Rightarrow cos\frac{\theta_C}{2} = 1 - \frac{2}{\pi}\sin^{-1}\left(sin\frac{\theta_H}{2}sin\frac{\theta_V}{2}\right) \Rightarrow \theta_C = 2\cos^{-1}\left\{\frac{\pi - 2\sin^{-1}\left(sin\frac{\theta_H}{2}sin\frac{\theta_V}{2}\right)}{\pi}\right\}$$

Thus **for a given uniform point-source, a beam having rectangular profile with horizontal & vertical beam angles $\boldsymbol{\theta_H}$ & $\boldsymbol{\theta_V}$, is equivalent to a beam having circular profile with a cone angle $\boldsymbol{\theta_C}$** given as

$$\boldsymbol{\theta_C = 2\cos^{-1}\left\{\frac{\pi - 2\sin^{-1}\left(sin\frac{\theta_H}{2}sin\frac{\theta_V}{2}\right)}{\pi}\right\}} \qquad (\forall \quad \boldsymbol{\theta_H, \theta_V} \in [\boldsymbol{0, \pi}])$$

If $\boldsymbol{\pi \le \theta_H, \theta_V \le 2\pi}$ then we have

$$2\pi\left(1 - cos\frac{\theta_C}{2}\right) = 4\pi - 4\sin^{-1}\left(sin\frac{\theta_H}{2}sin\frac{\theta_V}{2}\right)$$

$$\Rightarrow cos\frac{\theta_C}{2} = 2 - \frac{2}{\pi}\sin^{-1}\left(sin\frac{\theta_H}{2}sin\frac{\theta_V}{2}\right) \Rightarrow \theta_C = 2\cos^{-1}\left\{\frac{2\pi - 2\sin^{-1}\left(sin\frac{\theta_H}{2}sin\frac{\theta_V}{2}\right)}{\pi}\right\}$$

$$\boldsymbol{\theta_C = 2\cos^{-1}\left\{\frac{2\pi - 2\sin^{-1}\left(sin\frac{\theta_H}{2}sin\frac{\theta_V}{2}\right)}{\pi}\right\}} \qquad (\forall \quad \boldsymbol{\theta_H, \theta_V} \in [\boldsymbol{\pi, 2\pi}])$$

2. Solid angle subtended by the beam with rectangular profile given lateral beam angles $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$

Let there be a beam with rectangular profile emitted from a uniform point-source S such that **$\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$ are the lateral beam angles** measured between the adjacent lateral beam rays (passing through the vertices of rectangular profile) originated from the point-source S. Now, consider an imaginary rectangular plane ABCD with centre O, length l & width b at a normal distance $OS = h$ from the uniform point-source S (As shown in the figure 2 above)

In right ΔANS (see fig-2 above)

$$tan\angle ASN = \frac{AN}{SN} \quad \Rightarrow tan\frac{\angle ASB}{2} = \frac{AN}{\sqrt{(OS)^2 + (ON)^2}} \quad (setting\ the\ value\ of\ SN\ from\ right\ \Delta NOS)$$

$$\Rightarrow tan\frac{\alpha}{2} = \frac{\frac{l}{2}}{\sqrt{h^2 + \left(\frac{b}{2}\right)^2}} \quad \Rightarrow \sqrt{\boldsymbol{b^2 + 4h^2}} = \boldsymbol{l\,cot\frac{\alpha}{2}}$$

Similarly, in right ΔDPS (see fig-2 above)

$$tan\angle DSP = \frac{DP}{SP} \quad \Rightarrow tan\frac{\angle ASD}{2} = \frac{DP}{\sqrt{(OS)^2 + (OP)^2}} \quad (setting\ the\ value\ of\ SP\ from\ right\ \Delta POS)$$

$$\Rightarrow tan\frac{\beta}{2} = \frac{\frac{b}{2}}{\sqrt{h^2 + \left(\frac{l}{2}\right)^2}} \quad \Rightarrow \sqrt{\boldsymbol{l^2 + 4h^2}} = \boldsymbol{b\,cot\frac{\beta}{2}}$$

Now, the solid angle (ω) subtended by the rectangular plane ABCD, having length l & width b lying at a normal distance $OS = h$, at the point-source S is given by the **standard formula of rectangular plane** as follows

$$\boldsymbol{\omega = 4\sin^{-1}\left(\frac{lb}{\sqrt{(l^2 + 4h^2)(b^2 + 4h^2)}}\right)} \qquad (as\ derived\ in\ \boldsymbol{HCR's\ Theory\ of\ Polygon})$$

Now, substituting the values of $\sqrt{l^2 + 4h^2}$ & $\sqrt{b^2 + 4h^2}$ in the above expression, we get

$$\omega = 4\sin^{-1}\left(\frac{lb}{\left(l\,cot\frac{\alpha}{2}\right)\left(b\,cot\frac{\beta}{2}\right)}\right) = 4\sin^{-1}\left(tan\frac{\alpha}{2}tan\frac{\beta}{2}\right)$$

Hence, the **solid angle ($\boldsymbol{\omega}$) subtended by a beam with rectangular profile at the uniform point-source given the lateral beam angles $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$**, is given as

$$\boldsymbol{\omega = 4\sin^{-1}\left(tan\frac{\alpha}{2}tan\frac{\beta}{2}\right)} \qquad (\forall \quad \boldsymbol{\alpha, \beta \in [0, \pi)})$$

Surface area intercepted by the beam with a spherical surface: The surface area intercepted by the beam, having rectangular profile with lateral beam angles α & β, with the spherical surface having a radius R & centre at the point-source is given as

$$A_S = (solid\ angle) \times (radius\ of\ spherical\ surface)^2 = \omega R^2 = 4R^2\sin^{-1}\left(tan\frac{\alpha}{2}tan\frac{\beta}{2}\right)$$

$$\boldsymbol{Spherical\ surface\ area\ intercepted\ by\ the\ beam,\ \ A_S = 4R^2\sin^{-1}\left(tan\frac{\alpha}{2}tan\frac{\beta}{2}\right)}$$

Beam with a circular profile equivalent to the beam with a rectangular profile emitted from the same uniform point-source: Let's consider a beam with circular profile subtending a cone angle θ_C & a solid angle ω at the uniform point-source S (As shown in the figure 3 above)

Now, equating the values of solid angles subtended by the beams with circular & rectangular profiles at the given uniform point source S as follows

$$2\pi\left(1-\cos\frac{\theta_C}{2}\right)=4\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$$

$$\Rightarrow\ \cos\frac{\theta_C}{2}=1-\frac{2}{\pi}\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)\ \Rightarrow\ \theta_C=2\cos^{-1}\left\{\frac{\pi-2\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)}{\pi}\right\}$$

Thus **for a given uniform point-source, a beam having rectangular profile with the lateral beam angles $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$, is equivalent to a beam having circular profile with cone angle $\boldsymbol{\theta_C}$** given as

$$\boldsymbol{\theta_C=2\cos^{-1}\left\{\frac{\pi-2\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)}{\pi}\right\}}$$

Conclusion: Thus, for the given values of horizontal & vertical beam angles θ_H & θ_V or the lateral beam angles α & β, we can easily calculate important parameters such as solid angle subtended by the beam at the point-source, total area intercepted by the beam with a spherical surface & cone angle of equivalent beam with circular section. These formulae are very useful for replacing the rectangular profile by circular profile of a beam emitted by a uniform point-source & vice versa without any change in the total radiation energy/luminous flux associated with the original beam (having either rectangular or circular profile). Thus the articles discussed & analysed here are very useful for the analysis of radiation energy & determining the intensity of (uniform) point-source in a given direction in Radiometry & for the analysis of luminous flux & luminous intensity of (uniform) point-source in a certain direction in Photometry. These are also very useful in replacing the rectangular aperture by a circular aperture & vice versa without any change in the total radiation energy/luminous flux passing through an original aperture & thus useful for the analysis & the designing of optical\beam emitting (like laser) devices based on (uniform) point-sources.

Note: *Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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