$Mathematical\ analysis\ of\ identical\ circles\ touching\ one\ another\ on\ the\ spherical\ polyhedrons\ analogous\ to\ Archimedean\ solids$

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Introduction: Here, we are to analyse the identical circles, each having a flat radius r, touching one another on the **spherical polyhedrons** (i.e. **tiled spherical surfaces**), having a radius R, analogous to all 13 Archimedean solids by using the **tables (of all Archimedean solids prepared by the author)** for determining the following

1. **Relation between $\boldsymbol{R}$ & $\boldsymbol{r}$**
2. **Radius of each circle as a great circle arc** $(\boldsymbol{arc\ r})$ &
3. **Total surface area & its percentage** $(\%)$ covered by all the circles on the spherical polyhedron.

For such cases, each of certain no. of the identical circles is assumed to be centred at each of the identical vertices of a spherical polyhedron analogous to an Archimedean solid. Thus, each of the identical circles touches other circles exactly at the mid-points of great circle arcs representing the edges of all faces of analogous Archimedean solid (as shown in the figure 1). Then for a given spherical polyhedron analogous to an Archimedean solid, we have

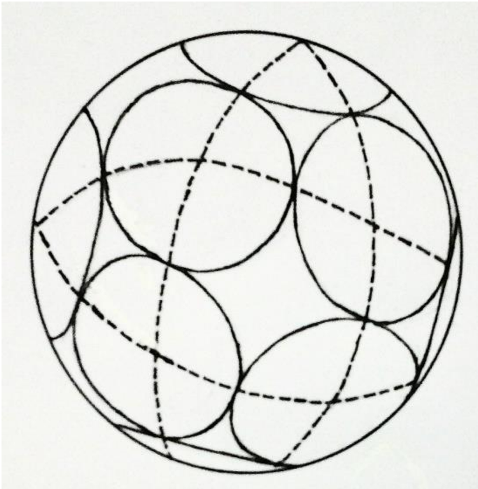

Figure 1: Certain no. of the identical circles, touching one-another, centred at the vertices of a spherical polyhedron analogous to an Archimedean solid

$$\boldsymbol{Total\ no.of\ identical\ circles\ touching\ one\ another = no.of\ vertices\ of\ analogous\ polyhedron}$$

$$\boldsymbol{No.of\ points\ at\ which\ each\ circle\ touches\ others\ = no.of\ edges\ meeting\ at\ each\ vertex\ of\ analogous\ polyhedron}$$

$$\boldsymbol{Total\ no.of\ points\ of\ tangency = no.of\ edges\ of\ analogous\ polyhedron}$$

Now, consider a **spherical polyhedron** (**tiled sphere**), with a **radius $\boldsymbol{R}$** & centre O, analogous to an **Archimedean solid** having **edge length $\boldsymbol{a}$** & centre O (coincident with the centre of spherical polyhedron). Draw $\boldsymbol{n}$ no. of the identical circles each having a **flat (plane) radius $\boldsymbol{r}$** (or arc radius $\boldsymbol{arc\ r}$) & centred at one of the vertices of the given spherical polyhedron such that they touch one another exactly at the mid-points of all the great circle arcs representing all the edges of the analogous Archimedean solid. Now, consider any of great circle arcs say AB representing (straight) edge $AB = a$ of Archimedean solid. The points A & B are the centres of two identical circles touching each other at the mid-point C of a great circle arc AB. Join the points A, B & C to the centre of the spherical polyhedron (See the figure 2). We have

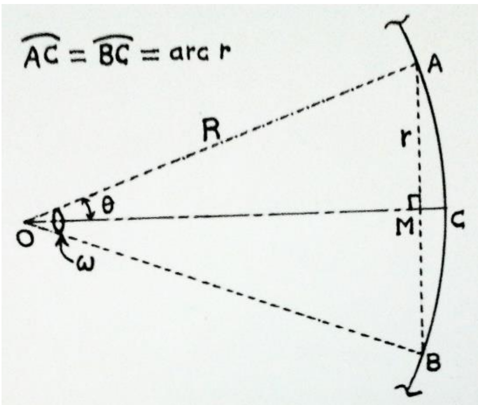

Figure 2: Great circle arc AB on a spherical polyhedron represents the (straight) edge AB of analogous Archimedean solid with centre O.

$$AM = BM = \frac{AB}{2} = \frac{a}{2} = r$$

$$\therefore\ \boldsymbol{Flat\ radius\ of\ each\ circle},\quad \boldsymbol{r = \frac{a}{2}}$$

In right ΔAMO

$$sin\theta = \frac{AM}{OA} = \frac{\frac{a}{2}}{R} = \frac{a}{2R} \quad \Rightarrow \boldsymbol{\theta = \sin^{-1}\left(\frac{a}{2R}\right)}$$

$$\Rightarrow arc\ BC = arc\ AC = R\theta = R\sin^{-1}\left(\frac{a}{2R}\right) = arc\ radius\ of\ circle$$

$$\boldsymbol{\therefore\ Arc\ radius\ of\ each\ circle,\ \ arc\ r = R\sin^{-1}\left(\frac{a}{2R}\right)}$$

Total surface area (A_s) covered by all the identical circles on the spherical polyhedron as whole: In order to calculate the surface area covered by each of n no. of the identical circles on the spherical polyhedron with a radius R, let's first find out the solid angle subtended by each circle with a flat radius r at the centre O of the spherical surface (See the figure 2 above) by using **formula of solid angle of a right cone concentric with a sphere** as follows

$$Solid\ angle\ subtended\ by\ each\ circle\ at\ the\ centre\ of\ the\ spherical\ polyhedron, \omega = 2\pi(1 - cos\theta)$$

$$\Rightarrow \omega = 2\pi\left(1 - \sqrt{1 - sin^2\theta}\right)$$

$$= 2\pi\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right) \quad \left(setting\ the\ value\ of\ \sin\theta = \frac{a}{2R}\ from\ the\ figure\ 2\ above\right)$$

Hence, the **total surface area, covered by all n identical circles on the spherical polyhedron as whole**, is given as

$$A_s = (no.of\ circles) \times (solid\ angle\ (\omega)\ subtended\ by\ each\ circle) \times (R^2)$$

$$= n\left(2\pi\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right)\right)R^2 = 2n\pi R^2\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right)$$

$$\boldsymbol{\therefore Total\ surface\ area\ covered,\ \ A_s = 2n\pi R^2\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right)}$$

Hence, the **percentage of total surface area covered by all n identical circles on the spherical polyhedron**, is given as

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered = \frac{total\ surface\ area\ covered\ by\ all\ the\ circles}{total\ surface\ area\ of\ the\ sphere} \times 100}$$

$$= \frac{2n\pi R^2\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right)}{4\pi R^2} \times 100 = 50n\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right)\%$$

$$\boldsymbol{\therefore\ \%\ of\ total\ surface\ area\ covered = 50n\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right)\%}$$

Thus, by directly using the tables of Archimedean solids & above generalized expressions, we will analyse the spherical polyhedrons corresponding to all 13 Archimedean solids in an order as discussed below.

1.) 12 identical circles, each having a flat radius r, touching one another on the spherical truncated tetrahedron with a radius R (analogous to truncated tetrahedron): In this case**,** let's assume that each of 12 identical circles, with a flat radius r, is centred at each of 12 identical vertices of a spherical truncated tetrahedron, with a radius R, analogous to the truncated tetrahedron with edge length a. Then in this case, we have

$$Total\ no.of\ identical\ circles\ touching\ one\ another,$$
$$n = no.of\ vertices\ of\ analogous\ truncated\ tetrahedron = 12$$

$$No.of\ points\ at\ which\ each\ circle\ touches\ others$$
$$= no.of\ edges\ meeting\ at\ each\ vertex\ of\ analogous\ truncated\ tetrahedron = 3$$

$$Total\ no.of\ points\ of\ tangency = no.of\ edges\ of\ analogous\ truncated\ tetrahedron = 18$$

From the **table of truncated tetrahedron**, the radius R of spherical truncated tetrahedron & edge length a are co-related as follows (refer to the table for **value of outer (circumscribed) radius of truncated tetrahedron**)

$$R = \frac{a}{2}\sqrt{\frac{11}{2}} \quad \Rightarrow \quad \boxed{\boldsymbol{a = 2R\sqrt{\frac{2}{11}}}}$$

Hence by setting the value of a in term of R, all the important parameters are calculated as follows

$$\boldsymbol{Flat\ radius\ of\ each\ circle,}\quad \boldsymbol{r} = \frac{a}{2} = \boldsymbol{R\sqrt{\frac{2}{11}} \approx 0.426401432\ R}$$

$$\boldsymbol{Arc\ radius\ of\ each\ circle,}\quad \boldsymbol{arc\ r} = R\sin^{-1}\left(\frac{a}{2R}\right) = \boldsymbol{R\sin^{-1}\left(\sqrt{\frac{2}{11}}\right) \approx 0.440510663\ R}$$

$$\boldsymbol{Total\ surface\ area\ covered,}\quad \boldsymbol{A_s} = 2n\pi R^2\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right) = 2(12)\pi R^2\left(1-\sqrt{1-\left(\sqrt{\frac{2}{11}}\right)^2}\right)$$

$$= 24\pi R^2\left(1-\sqrt{\frac{9}{11}}\right) = \boldsymbol{24\pi R^2\left(1-\frac{3}{\sqrt{11}}\right) \approx 7.197964279\ R^2}$$

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = 50n\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right) = 50(12)\left(1-\sqrt{1-\left(\sqrt{\frac{2}{11}}\right)^2}\right)$$

$$= 600\left(1-\sqrt{\frac{9}{11}}\right) = \boldsymbol{600\left(1-\frac{3}{\sqrt{11}}\right)}\ \% \approx \boldsymbol{57.28}\ \%$$

KEY POINT-1: 12 identical circles, touching one another at 18 different points (i.e. each one touches three other circles), on a spherical truncated tetrahedron, always cover up approximately $57.28\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $42.72\ \%$ of total surface area is left uncovered by the circles.

2.) 24 identical circles, each having a flat radius r, touching one another on the spherical truncated cube with a radius R (analogous to truncated cube): In this case**,** let's assume that each of 24 identical circles, with a flat radius r, is centred at each of 24 identical vertices of a spherical truncated cube, with a radius R, analogous to the truncated cube with edge length a. Then in this case, we have

$$\text{Total no. of identical circles touching one another,}$$
$$n = \text{no. of vertices of analogous truncated cube} = 24$$

$$\text{No. of points at which each circle touches others}$$
$$= \text{no. of edges meeting at each vertex of analogous truncated cube} = 3$$

$$\text{Total no. of points of tangency} = \text{no. of edges of analogous truncated cube} = 36$$

From the **table of truncated cube**, the radius R of spherical truncated cube & edge length a are co-related as follows (refer to the table for the **value of outer (circumscribed) radius of truncated cube**)

$$R = \frac{a}{2}\sqrt{7+4\sqrt{2}} \quad \Rightarrow \quad \boldsymbol{a = 2R\sqrt{\frac{7-4\sqrt{2}}{17}}}$$

Hence by setting the value of a in term of R, all the important parameters are calculated as follows

$$\textbf{\textit{Flat radius of each circle}},\quad r = \frac{a}{2} = \boldsymbol{R\sqrt{\frac{7-4\sqrt{2}}{17}} \approx 0.281084637\,R}$$

$$\textbf{\textit{Arc radius of each circle}},\quad \boldsymbol{arc\ r} = R\sin^{-1}\left(\frac{a}{2R}\right) = \boldsymbol{R\sin^{-1}\left(\sqrt{\frac{7-4\sqrt{2}}{17}}\right) \approx 0.284924126\,R}$$

$$\textbf{\textit{Total surface area covered}},\quad \boldsymbol{A_s} = 2n\pi R^2\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right)$$

$$= 2(24)\pi R^2\left(1-\sqrt{1-\left(\sqrt{\frac{7-4\sqrt{2}}{17}}\right)^2}\right) = \boldsymbol{48\pi R^2\left(1-\sqrt{\frac{10+4\sqrt{2}}{17}}\right) \approx 6.079663044\,R^2}$$

$$\%\ \textbf{\textit{of total surface area covered}} = 50n\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right) = 50(24)\left(1-\sqrt{1-\left(\sqrt{\frac{7-4\sqrt{2}}{17}}\right)^2}\right)$$

$$= \boldsymbol{1200\left(1-\sqrt{\frac{10+4\sqrt{2}}{17}}\right)}\ \% \approx \boldsymbol{48.38}\ \%$$

KEY POINT-2: 24 identical circles, touching one another at 36 different points (i.e. each one touches three other circles), on a spherical truncated hexahedron (cube), always cover up approximately $48.38\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $51.62\ \%$ of total surface area is left uncovered by the circles.

3.) 24 identical circles, each having a flat radius r, touching one another on the spherical truncated octahedron with a radius R (analogous to truncated octahedron): In this case**,** let's assume that each of 24 identical circles, with a flat radius r, is centred at each of 24 identical vertices of a spherical truncated octahedron, with a radius R, analogous to the truncated octahedron with edge length a. Then in this case, we have

$$Total\ no.of\ identical\ circles\ touching\ one\ another,$$
$$n = no.of\ vertices\ of\ analogous\ truncated\ octahedron = 24$$

$$No.of\ points\ at\ which\ each\ circle\ touches\ others$$
$$= no.of\ edges\ meeting\ at\ each\ vertex\ of\ analogous\ truncated\ octahedron = 3$$

$$Total\ no.of\ points\ of\ tangency = no.of\ edges\ of\ analogous\ truncated\ octahedron = 36$$

From the **table of truncated octahedron**, the radius R of spherical truncated octahedron & edge length a are co-related as follows (refer to the table for the **value of outer (circumscribed) radius of truncated octahedron**)

$$R = a\sqrt{\frac{5}{2}} \quad \Rightarrow \quad \boxed{\boldsymbol{a = R\sqrt{\frac{2}{5}}}}$$

Hence by setting the value of a in term of R, all the important parameters are calculated as follows

$$\boldsymbol{Flat\ radius\ of\ each\ circle},\ \ r = \frac{a}{2} = \frac{\boldsymbol{R}}{\boldsymbol{\sqrt{10}}} \approx \boldsymbol{0.316227766\ R}$$

$$\boldsymbol{Arc\ radius\ of\ each\ circle},\ \ \boldsymbol{arc\ r} = R\sin^{-1}\left(\frac{a}{2R}\right) = \boldsymbol{R\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)} \approx \boldsymbol{0.321750554\ R}$$

$$\boldsymbol{Total\ surface\ area\ covered},\ \ \boldsymbol{A_s} = 2n\pi R^2\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right)$$

$$= 2(24)\pi R^2\left(1 - \sqrt{1 - \left(\frac{1}{\sqrt{10}}\right)^2}\right) = 48\pi R^2\left(1 - \sqrt{\frac{9}{10}}\right) = \boldsymbol{48\pi R^2\left(1 - \frac{3}{\sqrt{10}}\right)} \approx \boldsymbol{7.738376345\ R^2}$$

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = 50n\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right) = 50(24)\left(1 - \sqrt{1 - \left(\frac{1}{\sqrt{10}}\right)^2}\right)$$

$$= 1200\left(1 - \sqrt{\frac{9}{10}}\right) = \boldsymbol{1200\left(1 - \frac{3}{\sqrt{10}}\right)}\ \% \approx \boldsymbol{61.58}\ \%$$

KEY POINT-3: 24 identical circles, touching one another at 36 different points (i.e. each one touches three other circles), on a spherical truncated octahedron, always cover up approximately $61.58\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $38.42\ \%$ of total surface area is left uncovered by the circles.

4.) 60 identical circles, each having a flat radius r, touching one another on the spherical truncated dodecahedron with a radius R (analogous to truncated dodecahedron): In this case**,** let's assume that each of 60 identical circles, with a flat radius r, is centred at each of 60 identical vertices of a spherical truncated dodecahedron, with a radius R, analogous to the truncated dodecahedron with edge length a. Then in this case, we have

$$\textit{Total no. of identical circles touching one another},\quad n = \textit{no. of vertices of analogous truncated dodecahedron} = 60$$

$$\textit{No. of points at which each circle touches others} = \textit{no. of edges meeting at each vertex of analogous truncated dodecahedron} = 3$$

$$\textit{Total no. of points of tangency} = \textit{no. of edges of analogous truncated dodecahedron} = 90$$

From the **table of truncated dodecahedron**, the radius R of spherical truncated dodecahedron & edge length a are co-related as follows (refer to the table for the **value of outer (circumscribed) radius of truncated dodecahedron**)

$$R = \frac{a}{4}\sqrt{74+30\sqrt{5}} \quad \Rightarrow \quad \boxed{\boldsymbol{a = 2R\sqrt{\frac{37-15\sqrt{5}}{122}}}}$$

Hence by setting the value of a in term of R, all the important parameters are calculated as follows

$$\textit{Flat radius of each circle},\quad r = \frac{a}{2} = \boldsymbol{R\sqrt{\frac{37-15\sqrt{5}}{122}} \approx 0.168381405\, R}$$

$$\textit{Arc radius of each circle},\quad \textit{arc } r = R\sin^{-1}\left(\frac{a}{2R}\right) = \boldsymbol{R\sin^{-1}\left(\sqrt{\frac{37-15\sqrt{5}}{122}}\right) \approx 0.169187398\, R}$$

$$\textit{Total surface area covered},\quad A_s = 2n\pi R^2\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right)$$

$$= 2(60)\pi R^2\left(1-\sqrt{1-\left(\sqrt{\frac{37-15\sqrt{5}}{122}}\right)^2}\right) = \boldsymbol{120\pi R^2\left(1-\sqrt{\frac{85+15\sqrt{5}}{122}}\right) \approx 5.382709618\, R^2}$$

$$\%\ \textit{of total surface area covered} = 50n\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right) = 50(60)\left(1-\sqrt{1-\left(\sqrt{\frac{37-15\sqrt{5}}{122}}\right)^2}\right)$$

$$= \boldsymbol{3000\left(1-\sqrt{\frac{85+15\sqrt{5}}{122}}\right)\% \approx 42.83\ \%}$$

KEY POINT-4: 60 identical circles, touching one another at 90 different points (i.e. each one touches three other circles), on a spherical truncated dodecahedron, always cover up approximately $42.83\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $57.17\ \%$ of total surface area is left uncovered by the circles.

5.) 60 identical circles, each having a flat radius r, touching one another on the spherical truncated icosahedron with a radius R (analogous to truncated icosahedron): In this case**,** let's assume that each of 60 identical circles, with a flat radius r, is centred at each of 60 identical vertices of a spherical truncated icosahedron, with a radius R, analogous to the truncated icosahedron with edge length a. Then in this case, we have

$$Total\ no.of\ identical\ circles\ touching\ one\ another,$$
$$n = no.of\ vertices\ of\ analogous\ truncated\ icosahedron = 60$$

$$No.of\ points\ at\ which\ each\ circle\ touches\ others$$
$$= no.of\ edges\ meeting\ at\ each\ vertex\ of\ analogous\ truncated\ icosahedron = 3$$

$$Total\ no.of\ points\ of\ tangency = no.of\ edges\ of\ analogous\ truncated\ icosahedron = 90$$

From the **table of truncated icosahedron**, the radius R of spherical truncated icosahedron & edge length a are co-related as follows (refer to the table for the **value of outer (circumscribed) radius of truncated icosahedron**)

$$R = \frac{a}{4}\sqrt{58 + 18\sqrt{5}} \quad \Rightarrow \quad \boxed{\boldsymbol{a = 2R\sqrt{\frac{29 - 9\sqrt{5}}{218}}}}$$

Hence by setting the value of a in term of R, all the important parameters are calculated as follows

$$\boldsymbol{Flat\ radius\ of\ each\ circle},\quad r = \frac{a}{2} = \boldsymbol{R\sqrt{\frac{29 - 9\sqrt{5}}{218}} \approx 0.201774106\ R}$$

$$\boldsymbol{Arc\ radius\ of\ each\ circle},\quad \boldsymbol{arc\ \ r} = R\sin^{-1}\left(\frac{a}{2R}\right) = \boldsymbol{R\sin^{-1}\left(\sqrt{\frac{29 - 9\sqrt{5}}{218}}\right) \approx 0.203168946\ R}$$

$$\boldsymbol{Total\ surface\ area\ covered},\quad \boldsymbol{A_s} = 2n\pi R^2\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right)$$

$$= 2(60)\pi R^2\left(1 - \sqrt{1 - \left(\sqrt{\frac{29 - 9\sqrt{5}}{218}}\right)^2}\right) = \boldsymbol{120\pi R^2\left(1 - \sqrt{\frac{189 + 9\sqrt{5}}{218}}\right) \approx 7.753921096\ R^2}$$

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = 50n\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right) = 50(60)\left(1 - \sqrt{1 - \left(\sqrt{\frac{29 - 9\sqrt{5}}{218}}\right)^2}\right)$$

$$= \mathbf{3000}\left(\mathbf{1}-\sqrt{\frac{\mathbf{189}+\mathbf{9}\sqrt{\mathbf{5}}}{\mathbf{218}}}\right)\% \approx \mathbf{61.7}\,\%$$

KEY POINT-5: 60 identical circles, touching one another at 90 different points (i.e. each one touches three other circles), on a spherical truncated icosahedron, always cover up approximately $61.7\,\%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $38.3\,\%$ of total surface area is left uncovered by the circles.

6.) 12 identical circles, each having a flat radius r, touching one another on the spherical truncated cuboctahedron with a radius R (analogous to truncated cuboctahedron): In this case**,** let's assume that each of 12 identical circles, with a flat radius r, is centred at each of 12 identical vertices of a spherical truncated cuboctahedron, with a radius R, analogous to the truncated cuboctahedron with edge length a. Then in this case, we have

$$\mathit{Total\ no.of\ identical\ circles\ touching\ one\ another,}$$
$$n = \mathit{no.of\ vertices\ of\ analogous\ truncated\ cuboctahedron} = 12$$

$$\mathit{No.of\ points\ at\ which\ each\ circle\ touches\ others}$$
$$= \mathit{no.of\ edges\ meeting\ at\ each\ vertex\ of\ analogous\ truncated\ cuboctahedron} = 4$$

$$\mathit{Total\ no.of\ points\ of\ tangency} = \mathit{no.of\ edges\ of\ analogous\ truncated\ cuboctahedron} = 24$$

From the **table of truncated cuboctahedron**, the radius R of spherical truncated cuboctahedron & edge length a are co-related as follows (refer to the table for the **value of outer (circumscribed) radius of truncated cuboctahedron**)

$$R = a \quad \Rightarrow \quad \boxed{\boldsymbol{a = R}}$$

Hence by setting the value of a in term of R, all the important parameters are calculated as follows

$$\boldsymbol{Flat\ radius\ of\ each\ circle,}\quad r = \frac{a}{2} = \frac{\boldsymbol{R}}{\mathbf{2}} = \mathbf{0.5}\,\boldsymbol{R}$$

$$\boldsymbol{Arc\ radius\ of\ each\ circle,}\quad \boldsymbol{arc\ r} = R\sin^{-1}\left(\frac{a}{2R}\right) = \boldsymbol{R}\sin^{-1}\left(\frac{\mathbf{1}}{\mathbf{2}}\right) = \frac{\boldsymbol{\pi R}}{\mathbf{6}} \approx \mathbf{0.523598775}\,\boldsymbol{R}$$

$$\boldsymbol{Total\ surface\ area\ covered,}\quad \boldsymbol{A_s} = 2n\pi R^2\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right)$$

$$= 2(12)\pi R^2\left(1-\sqrt{1-\left(\frac{1}{2}\right)^2}\right) = \mathbf{24}\boldsymbol{\pi R^2}\left(\mathbf{1}-\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\right) \approx \mathbf{10.10144657}\,\boldsymbol{R^2}$$

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = 50n\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right) = 50(12)\left(1-\sqrt{1-\left(\frac{1}{2}\right)^2}\right)$$

$$= \mathbf{600}\left(\mathbf{1}-\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\right)\% \approx \mathbf{80.38}\,\%$$

KEY POINT-6: 12 identical circles, touching one another at 24 different points (i.e. each one touches four other circles), on a spherical truncated cuboctahedron, always cover up approximately $80.38\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $19.62\ \%$ of total surface area is left uncovered by the circles.

7.) 30 identical circles, each having a flat radius r, touching one another on the spherical truncated icosidodecahedron with a radius R (analogous to truncated icosidodecahedron): In this case**,** let's assume that each of 30 identical circles, with a flat radius r, is centred at each of 30 identical vertices of a spherical truncated icosidodecahedron, with a radius R, analogous to the truncated icosidodecahedron with edge length a. Then in this case, we have

$$Total\ no.of\ identical\ circles\ touching\ one\ another,$$
$$n = no.of\ vertices\ of\ analogous\ truncated\ icosidodecahedron = 30$$

$$No.of\ points\ at\ which\ each\ circle\ touches\ others$$
$$= no.of\ edges\ meeting\ at\ each\ vertex\ of\ analog\ truncated\ icosidodecahedron = 4$$

$$Total\ no.of\ points\ of\ tangency = no.of\ edges\ of\ analogous\ truncated\ icosidodecahedron = 60$$

From the **table of truncated icosidodecahedron**, the radius R of spherical truncated icosidodecahedron & edge length a are co-related as follows (refer to the table for the **value of outer (circumscribed) radius of truncated icosidodecahedron**)

$$R = \frac{a(\sqrt{5}+1)}{2} \quad \Rightarrow \quad \boxed{\boldsymbol{a = \frac{R(\sqrt{5}-1)}{2}}}$$

Hence by setting the value of a in term of R, all the important parameters are calculated as follows

$$\boldsymbol{Flat\ radius\ of\ each\ circle},\quad r = \frac{a}{2} = \boldsymbol{\frac{R(\sqrt{5}-1)}{4} \approx 0.309016994\ R}$$

$$\boldsymbol{Arc\ radius\ of\ each\ circle},\quad \boldsymbol{arc\ r} = R\sin^{-1}\left(\frac{a}{2R}\right) = \boldsymbol{R\sin^{-1}\left(\frac{\sqrt{5}-1}{4}\right) \approx 0.314159265\ R}$$

$$\boldsymbol{Total\ surface\ area\ covered},\quad \boldsymbol{A_s} = 2n\pi R^2\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right)$$

$$= 2(30)\pi R^2\left(1-\sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2}\right) = \boldsymbol{60\pi R^2\left(1-\frac{\sqrt{10+2\sqrt{5}}}{4}\right) \approx 9.225629331\ R^2}$$

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = 50n\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right) = 50(30)\left(1-\sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2}\right)$$

$$= \boldsymbol{1500\left(1-\frac{\sqrt{10+2\sqrt{5}}}{4}\right)}\ \% \approx \boldsymbol{73.41}\ \%$$

KEY POINT-7: 30 identical circles, touching one another at 60 different points (i.e. each one touches four other circles), on a spherical truncated icosidodecahedron, always cover up approximately $73.41\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $26.59\ \%$ of total surface area is left uncovered by the circles.

8.) 24 identical circles, each having a flat radius r, touching one another on the spherical snub cube with a radius R (analogous to snub cube): In this case**,** let's assume that each of 24 identical circles, with a flat radius r, is centred at each of 24 identical vertices of a spherical snub cube, with a radius R, analogous to the snub cube with edge length a. Then in this case, we have

$$Total\ no.of\ identical\ circles\ touching\ one\ another,$$
$$n = no.of\ vertices\ of\ analogous\ snub\ cube = 24$$

$$No.of\ points\ at\ which\ each\ circle\ touches\ others$$
$$= no.of\ edges\ meeting\ at\ each\ vertex\ of\ analogous\ snub\ cube = 5$$

$$Total\ no.of\ points\ of\ tangency = no.of\ edges\ of\ analogous\ snub\ cube = 60$$

From the **table of snub cube**, the radius R of spherical snub cube & edge length a are co-related as follows (refer to the table for the **value of outer (circumscribed) radius of snub cube**)

$$R = aC \quad \Rightarrow \quad \boldsymbol{a = \frac{R}{C} \quad \forall\ C \approx 1.343713374}$$

Hence by setting the value of a in term of R, all the important parameters are calculated as follows

$$\boldsymbol{Flat\ radius\ of\ each\ circle},\quad r = \frac{a}{2} = \boldsymbol{\frac{R}{2C}} \approx \boldsymbol{0.372103165\ R}$$

$$\boldsymbol{Arc\ radius\ of\ each\ circle},\quad \boldsymbol{arc\ r} = R\sin^{-1}\left(\frac{a}{2R}\right) = \boldsymbol{R\sin^{-1}\left(\frac{1}{2C}\right)} \approx \boldsymbol{0.381273869\ R}$$

$$\boldsymbol{Total\ surface\ area\ covered},\quad \boldsymbol{A_s} = 2n\pi R^2\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right)$$

$$= 2(24)\pi R^2\left(1 - \sqrt{1 - \left(\frac{1}{2C}\right)^2}\right) = \boldsymbol{48\pi R^2\left(1 - \frac{\sqrt{4C^2 - 1}}{2C}\right)} \approx \boldsymbol{10.82848509\ R^2}$$

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = 50n\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right) = 50(24)\left(1 - \sqrt{1 - \left(\frac{1}{2C}\right)^2}\right)$$

$$= \boldsymbol{1200\left(1 - \frac{\sqrt{4C^2 - 1}}{2C}\right)}\ \% \approx \boldsymbol{86.17}\ \%$$

KEY POINT-8: 24 identical circles, touching one another at 60 different points (i.e. each one touches five other circles), on a spherical snub cube, always cover up approximately $86.17\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $13.83\ \%$ of total surface area is left uncovered by the circles.

9.) 60 identical circles, each having a flat radius r, touching one another on the spherical snub dodecahedron with a radius R (analogous to snub dodecahedron): In this case**,** let's assume that each of 60 identical circles, with a flat radius r, is centred at each of 60 identical vertices of a spherical snub dodecahedron, with a radius R, analogous to the snub dodecahedron with edge length a. Then in this case, we have

$$\text{Total no. of identical circles touching one another,}$$
$$n = \text{no. of vertices of analogous snub dodecahedron} = 60$$

$$\text{No. of points at which each circle touches others}$$
$$= \text{no. of edges meeting at each vertex of analogous snub dodecahedron} = 5$$

$$\text{Total no. of points of tangency} = \text{no. of edges of analogous snub dodecahedron} = 150$$

From the **table of snub dodecahedron**, the radius R of spherical snub dodecahedron & edge length a are co-related as follows (refer to the table for the **value of outer (circumscribed) radius of snub dodecahedron**)

$$R = aC \quad \Rightarrow \quad \boldsymbol{a = \frac{R}{C} \quad \forall \; C \approx 2.155837375}$$

Hence by setting the value of a in term of R, all the important parameters are calculated as follows

$$\textbf{Flat radius of each circle}, \quad r = \frac{a}{2} = \frac{\boldsymbol{R}}{\boldsymbol{2C}} \approx \boldsymbol{0.23192844\, R}$$

$$\textbf{Arc radius of each circle}, \quad \boldsymbol{arc\ r} = R\sin^{-1}\left(\frac{a}{2R}\right) = \boldsymbol{R\sin^{-1}\left(\frac{1}{2C}\right) \approx 0.234059713\, R}$$

$$\textbf{Total surface area covered}, \quad A_s = 2n\pi R^2\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right)$$

$$= 2(60)\pi R^2\left(1 - \sqrt{1 - \left(\frac{1}{2C}\right)^2}\right) = \boldsymbol{120\pi R^2\left(1 - \frac{\sqrt{4C^2 - 1}}{2C}\right) \approx 10.27947316\, R^2}$$

$$\%\ \textbf{of total surface area covered} = 50n\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right) = 50(60)\left(1 - \sqrt{1 - \left(\frac{1}{2C}\right)^2}\right)$$

$$= \boldsymbol{3000\left(1 - \frac{\sqrt{4C^2 - 1}}{2C}\right)}\ \% \approx \boldsymbol{81.8}\ \%$$

KEY POINT-9: 60 identical circles, touching one another at 150 different points (i.e. each one touches five other circles), on a spherical snub dodecahedron, always cover up approximately $81.8\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $18.2\ \%$ of total surface area is left uncovered by the circles.

10.) 24 identical circles, each having a flat radius r, touching one another on the spherical small rhombicuboctahedron with a radius R (analogous to small rhombicuboctahedron): In this case**,** let's assume that each of 24 identical circles, with a flat radius r, is centred at each of 24 identical vertices of a spherical small rhombicuboctahedron, with a radius R, analogous to the small rhombicuboctahedron with edge length a. Then in this case, we have

$$Total\ no.of\ identical\ circles\ touching\ one\ another,$$
$$n = no.of\ vertices\ of\ analogous\ small\ rhombicuboctahedron = 24$$

$$No.of\ points\ at\ which\ each\ circle\ touches\ others$$
$$= no.of\ edges\ meeting\ at\ each\ vertex\ of\ analog\ small\ rhombicuboctahedron = 4$$

$$Total\ no.of\ points\ of\ tangency = no.of\ edges\ of\ analogous\ small\ rhombicuboctahedron = 48$$

From the **table of small rhombicuboctahedron**, the radius R of spherical small rhombicuboctahedron & edge length a are co-related as follows (refer to the table for the **value of outer (circumscribed) radius of small rhombicuboctahedron**)

$$R = \frac{a}{2}\sqrt{5+2\sqrt{2}} \quad \Rightarrow \quad \boldsymbol{a = 2R\sqrt{\frac{5-2\sqrt{2}}{17}}}$$

Hence by setting the value of a in term of R, all the important parameters are calculated as follows

$$\boldsymbol{Flat\ radius\ of\ each\ circle,}\quad r = \frac{a}{2} = \boldsymbol{R\sqrt{\frac{5-2\sqrt{2}}{17}} \approx 0.357406744\ R}$$

$$\boldsymbol{Arc\ radius\ of\ each\ circle,}\quad \boldsymbol{arc\ r} = R\sin^{-1}\left(\frac{a}{2R}\right) = \boldsymbol{R\sin^{-1}\left(\sqrt{\frac{5-2\sqrt{2}}{17}}\right) \approx 0.365489756\ R}$$

$$\boldsymbol{Total\ surface\ area\ covered,}\quad \boldsymbol{A_s} = 2n\pi R^2\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right)$$

$$= 2(24)\pi R^2\left(1-\sqrt{1-\left(\sqrt{\frac{5-2\sqrt{2}}{17}}\right)^2}\right) = \boldsymbol{48\pi R^2\left(1-\sqrt{\frac{12+2\sqrt{2}}{17}}\right) \approx 9.960281616\ R^2}$$

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = 50n\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right) = 50(24)\left(1-\sqrt{1-\left(\sqrt{\frac{5-2\sqrt{2}}{17}}\right)^2}\right)$$

$$= \boldsymbol{1200\left(1-\sqrt{\frac{12+2\sqrt{2}}{17}}\right)}\ \% \approx \boldsymbol{79.26}\ \%$$

KEY POINT-10: 24 identical circles, touching one another at 48 different points (i.e. each one touches four other circles), on a spherical small rhombicuboctahedron, always cover up approximately $79.26\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $20.74\ \%$ of total surface area is left uncovered by the circles.

11.) 60 identical circles, each having a flat radius r, touching one another on the spherical small rhombicosidodecahedron with a radius R (analogous to small rhombicosidodecahedron): In this case**,** let's assume that each of 60 identical circles, with a flat radius r, is centred at each of 60 identical vertices of a spherical small rhombicosidodecahedron, with a radius R, analogous to the small rhombicosidodecahedron with edge length a. Then in this case, we have

$$Total\ no. of\ identical\ circles\ touching\ one\ another,$$
$$n = no. of\ vertices\ of\ analogous\ small\ rhombicosidodecahedron = 60$$

$$No. of\ points\ at\ which\ each\ circle\ touches\ others$$
$$= no. of\ edges\ meeting\ at\ each\ vertex\ of\ analogous\ small\ rhombicosidodecahedron$$
$$= 4$$

$$Total\ no. of\ points\ of\ tangency = no. of\ edges\ of\ analogous\ small\ rhombicosidodecahedron = 120$$

From the **table of small rhombicosidodecahedron**, the radius R of spherical small rhombicosidodecahedron & edge length a are co-related as follows (refer to the table for the **value of outer (circumscribed) radius of small rhombicosidodecahedron**)

$$R = \frac{a}{2}\sqrt{11+4\sqrt{5}} \quad \Rightarrow \quad \boldsymbol{a = 2R\sqrt{\frac{11-4\sqrt{5}}{41}}}$$

Hence by setting the value of a in term of R, all the important parameters are calculated as follows

$$\boldsymbol{Flat\ radius\ of\ each\ circle},\quad r = \frac{a}{2} = \boldsymbol{R\sqrt{\frac{11-4\sqrt{5}}{41}} \approx 0.223918979\,R}$$

$$\boldsymbol{Arc\ radius\ of\ each\ circle},\quad \boldsymbol{arc\ \ r} = R\sin^{-1}\left(\frac{a}{2R}\right) = \boldsymbol{R\sin^{-1}\left(\sqrt{\frac{11-4\sqrt{5}}{41}}\right) \approx 0.225833709\,R}$$

$$\boldsymbol{Total\ surface\ area\ covered},\quad \boldsymbol{A_s} = 2n\pi R^2\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right)$$

$$= 2(60)\pi R^2\left(1-\sqrt{1-\left(\sqrt{\frac{11-4\sqrt{5}}{41}}\right)^2}\right) = \boldsymbol{120\pi R^2\left(1-\sqrt{\frac{30+4\sqrt{5}}{41}}\right) \approx 9.572648061\,R^2}$$

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = 50n\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right) = 50(60)\left(1-\sqrt{1-\left(\sqrt{\frac{11-4\sqrt{5}}{41}}\right)^2}\right)$$

$$= \boldsymbol{3000\left(1-\sqrt{\frac{30+4\sqrt{5}}{41}}\right)}\ \% \approx \boldsymbol{76.18}\ \%$$

KEY POINT-11: 60 identical circles, touching one another at 120 different points (i.e. each one touches four other circles), on a spherical small rhombicosidodecahedron, always cover up approximately $76.18\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $23.82\ \%$ of total surface area is left uncovered by the circles.

12.) 48 identical circles, each having a flat radius r, touching one another on the spherical great rhombicuboctahedron with a radius R (analogous to great rhombicuboctahedron): In this case**,** let's assume that each of 48 identical circles, with a flat radius r, is centred at each of 48 identical vertices of a spherical great rhombicuboctahedron, with a radius R, analogous to the great rhombicuboctahedron with edge length a. Then in this case, we have

$$Total\ no.of\ identical\ circles\ touching\ one\ another,$$
$$n = no.of\ vertices\ of\ analogous\ great\ rhombicuboctahedron = 24$$

$$No.of\ points\ at\ which\ each\ circle\ touches\ others$$
$$= no.of\ edges\ meeting\ at\ each\ vertex\ of\ analog\ great\ rhombicuboctahedron = 3$$

$$Total\ no.of\ points\ of\ tangency = no.of\ edges\ of\ analogous\ great\ rhombicuboctahedron = 72$$

From the **table of great rhombicuboctahedron**, the radius R of spherical great rhombicuboctahedron & edge length a are co-related as follows (refer to the table for the **value of outer (circumscribed) radius of great rhombicuboctahedron**)

$$R = \frac{a}{2}\sqrt{13+6\sqrt{2}} \quad \Rightarrow \quad \boldsymbol{a = 2R\sqrt{\frac{13-6\sqrt{2}}{97}}}$$

Hence by setting the value of a in term of R, all the important parameters are calculated as follows

$$\boldsymbol{Flat\ radius\ of\ each\ circle},\quad r = \frac{a}{2} = \boldsymbol{R\sqrt{\frac{13-6\sqrt{2}}{97}} \approx 0.215739405\ R}$$

$$\boldsymbol{Arc\ radius\ of\ each\ circle},\quad \boldsymbol{arc\ r} = R\sin^{-1}\left(\frac{a}{2R}\right) = \boldsymbol{R\sin^{-1}\left(\sqrt{\frac{13-6\sqrt{2}}{97}}\right) \approx 0.217449004\ R}$$

$$\boldsymbol{Total\ surface\ area\ covered},\quad \boldsymbol{A_s} = 2n\pi R^2\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right)$$

$$= 2(48)\pi R^2\left(1-\sqrt{1-\left(\sqrt{\frac{13-6\sqrt{2}}{97}}\right)^2}\right) = \boldsymbol{96\pi R^2\left(1-\sqrt{\frac{84+6\sqrt{2}}{97}}\right) \approx 7.102218242\ R^2}$$

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = 50n\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right) = 50(48)\left(1-\sqrt{1-\left(\sqrt{\frac{13-6\sqrt{2}}{97}}\right)^2}\right)$$

$$= 2400\left(1 - \sqrt{\frac{84 + 6\sqrt{2}}{97}}\right)\% \approx 56.52\ \%$$

KEY POINT-12: 48 identical circles, touching one another at 72 different points (i.e. each one touches three other circles), on a spherical great rhombicuboctahedron, always cover up approximately $56.52\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $43.48\ \%$ of total surface area is left uncovered by the circles.

13.) 120 identical circles, each having a flat radius r, touching one another on the spherical great rhombicosidodecahedron with a radius R (analogous to great rhombicosidodecahedron): In this case**,** let's assume that each of 120 identical circles, with a flat radius r, is centred at each of 120 identical vertices of a spherical great rhombicosidodecahedron, with a radius R, analogous to the great rhombicosidodecahedron with edge length a. Then in this case, we have

$$Total\ no.of\ identical\ circles\ touching\ one\ another,$$
$$n = no.of\ vertices\ of\ analogous\ great\ rhombicosidodecahedron = 120$$

$$No.of\ points\ at\ which\ each\ circle\ touches\ others$$
$$= no.of\ edges\ meeting\ at\ each\ vertex\ of\ analogous\ great\ rhombicosidodecahedron = 3$$

$$Total\ no.of\ points\ of\ tangency = no.of\ edges\ of\ analogous\ great\ rhombicosidodecahedron = 180$$

From the **table of great rhombicosidodecahedron**, the radius R of spherical great rhombicosidodecahedron & edge length a are co-related as follows (refer to the table for the **value of outer (circumscribed) radius of great rhombicosidodecahedron/the largest Archimedean solid)**

$$R = \frac{a}{2}\sqrt{31 + 12\sqrt{5}} \quad \Rightarrow \quad \boxed{a = 2R\sqrt{\frac{31 - 12\sqrt{5}}{241}}}$$

Hence by setting the value of a in term of R, all the important parameters are calculated as follows

$$Flat\ radius\ of\ each\ circle,\quad r = \frac{a}{2} = R\sqrt{\frac{31 - 12\sqrt{5}}{241}} \approx 0.131496087\ R$$

$$Arc\ radius\ of\ each\ circle,\quad arc\ r = R\sin^{-1}\left(\frac{a}{2R}\right) = R\sin^{-1}\left(\sqrt{\frac{31 - 12\sqrt{5}}{241}}\right) \approx 0.131878021\ R$$

$$Total\ surface\ area\ covered,\quad A_s = 2n\pi R^2\left(1 - \sqrt{1 - \left(\frac{a}{2R}\right)^2}\right)$$

$$= 2(120)\pi R^2\left(1 - \sqrt{1 - \left(\sqrt{\frac{31 - 12\sqrt{5}}{241}}\right)^2}\right) = 240\pi R^2\left(1 - \sqrt{\frac{210 + 12\sqrt{5}}{241}}\right) \approx 6.547061843\ R^2$$

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = 50n\left(1-\sqrt{1-\left(\frac{a}{2R}\right)^2}\right)$$

$$= 50(120)\left(1-\sqrt{1-\left(\sqrt{\frac{31-12\sqrt{5}}{241}}\right)^2}\right)$$

$$= \mathbf{6000}\left(\mathbf{1}-\sqrt{\frac{\mathbf{210+12\sqrt{5}}}{\mathbf{241}}}\right)\% \approx \mathbf{52.1}\,\%$$

KEY POINT-13: 120 identical circles, touching one another at 180 different points (i.e. each one touches three other circles), on a spherical great rhombicosidodecahedron, always cover up approximately $52.1\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $47.9\ \%$ of total surface area is left uncovered by the circles.

Conclusion: The above formulae are applicable on a certain no. of the identical circles touching one another at different points, centred at the identical vertices of a spherical polyhedron analogous to an Archimedean solid for calculating the different parameters such as **flat radius & arc radius of each circle**, **total surface area covered** by all the circles, **percentage of surface area covered** etc. These formulae are very useful for tiling, packing the identical circles in different patterns & analysing the spherical surfaces analogous to all 13 Archimedean solids. Thus also useful in designing & modelling of tiled spherical surfaces.

Note: *Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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