$HCR's\ Cosine\ Formula$

$Angle\ between\ the\ chords\ of\ two\ great\ circles\ arcs$

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Introduction: **HCR's cosine formula** derived by the author is a **trigonometric relation** of four parameters/angles. **It is applied to find out the (plane) angle between the chords (i.e. straight lines joining the ends points) of any two great circle arcs meeting each other at a common end point at a given angle on a spherical surface of finite radius.** Two **great circle arcs AB** & **AC** are meeting (intersecting) each other at their common end point A at a given angle $\boldsymbol{\theta}$ such that $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$ are the angles subtended by the great circle arcs AB & AC respectively at the centre O of the sphere. The chords AB & AC also meet each other at the common end point A at a plane angle $\boldsymbol{\theta_p}$ (As shown in the figure 1). Here, we are interested to derive a mathematical formula which correlates four parametric angles $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta}$ & $\boldsymbol{\theta_p}$ and is used to find out any one of four angles when other three are known.

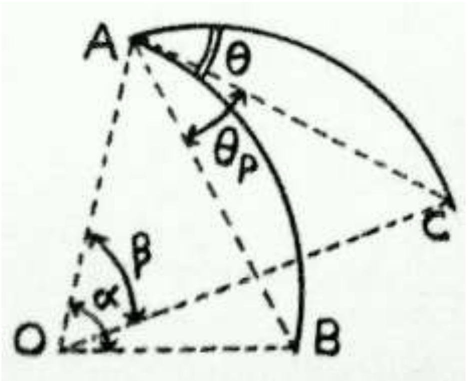

Figure 1: Two great circle arcs AB & AC meeting each other at a common end point A at an angle $\boldsymbol{\theta}$ & subtending angles $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$ at centre O

Derivation: Let there be any two **great circle arcs AB** & **AC** meeting (intersecting) each other at a common end point A at an angle $\boldsymbol{\theta}$ on a spherical surface of radius R. Let $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$ be the angles subtended by the great circle arcs AB & AC respectively at the centre O of the sphere and $\boldsymbol{\theta_p}$ be the angle between the corresponding chords AB & AC of the great circle arcs. (As shown in the figure 2)

Now, draw the tangents to the great circles arcs AB & AC at the common end point A & extend them so as to intersect the extended lines OB & OC at the points B' & C' respectively. The angle between the tangents AB' & AC' will be θ (equal to the angle between the great circle arcs AB & AC). Let $\angle B'OC' = \gamma$. Thus we obtain a tetrahedron OAB'C' having angles α, β & γ between consecutive edges meeting at the vertex O such that the angle between the lateral triangular faces OAB' & OAC' is θ. Now, using **inverse cosine formula** for unknown angle $\boldsymbol{\gamma}$ in the tetrahedron OAB'C' as follows

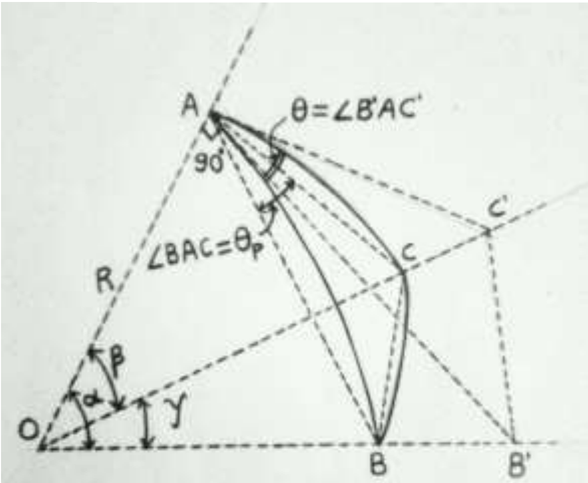

Figure 2: $\angle BAC = \boldsymbol{\theta_p}$ is the (plane) angle between the chords AB & AC of great circle arcs AB & AC which meet each at the common end point A

$$\boldsymbol{\gamma = \cos^{-1}(\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\theta)}$$

$$\cos\gamma = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\theta$$

$$1 - 2\sin^2\frac{\gamma}{2} = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\theta$$

$$2\sin^2\frac{\gamma}{2} = 1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta\cos\theta \quad \ldots\ldots\ldots\ldots\ldots(1)$$

In isosceles ΔAOB (see figure 2),

$$OA = OB = R,\ \angle AOB = \alpha$$

$$\sin\frac{\alpha}{2}=\frac{\frac{AB}{2}}{R}\quad\Rightarrow\ \boldsymbol{AB=2R\sin\frac{\alpha}{2}}$$

Similarly, in isosceles ΔAOC,

$$OA=OC=R,\ \angle AOC=\beta\ \ \Rightarrow\ \boldsymbol{AC=2R\sin\frac{\beta}{2}}$$

Similarly, in isosceles ΔBOC,

$$OB=OC=R,\ \angle BOC=\gamma\ \ \Rightarrow\ \boldsymbol{BC=2R\sin\frac{\gamma}{2}}$$

Now, join the end points A, B & C of the great circle arcs AB & AC by the dotted straight lines to get a plane ΔABC (As shown in the figure 2 above). **A**pplying **Cosine Rule** in plane ΔABC as follows

$$\cos\angle BAC=\frac{AB^2+AC^2-BC^2}{2(AB)(AC)}$$

Setting all the corresponding values in above expression, we get

$$\cos\theta_p=\frac{\left(2R\sin\frac{\alpha}{2}\right)^2+\left(2R\sin\frac{\beta}{2}\right)^2-\left(2R\sin\frac{\gamma}{2}\right)^2}{2\left(2R\sin\frac{\alpha}{2}\right)\left(2R\sin\frac{\beta}{2}\right)}$$

$$=\frac{2\sin^2\frac{\alpha}{2}+2\sin^2\frac{\beta}{2}-2\sin^2\frac{\gamma}{2}}{4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

Setting the value of $2\sin^2\frac{\gamma}{2}$ from eq(1), we get

$$\cos\theta_p=\frac{2\sin^2\frac{\alpha}{2}+2\sin^2\frac{\beta}{2}-(1-\cos\alpha\cos\beta-\sin\alpha\sin\beta\cos\theta)}{4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\theta+2\sin^2\frac{\alpha}{2}+2\sin^2\frac{\beta}{2}-1}{4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\theta+1-\left(1-2\sin^2\frac{\alpha}{2}\right)-\left(1-2\sin^2\frac{\beta}{2}\right)}{4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\theta+1-\cos\alpha-\cos\beta}{4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$=\frac{(1-\cos\alpha-\cos\beta+\cos\alpha\cos\beta)+\sin\alpha\sin\beta\cos\theta}{4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$=\frac{(1-\cos\alpha)(1-\cos\beta)+\sin\alpha\sin\beta\cos\theta}{4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$=\frac{\left(1-1+2sin^2\frac{\alpha}{2}\right)\left(1-1+2sin^2\frac{\beta}{2}\right)+\left(2sin\frac{\alpha}{2}cos\frac{\alpha}{2}\right)\left(2sin\frac{\beta}{2}cos\frac{\beta}{2}\right)cos\theta}{4sin\frac{\alpha}{2}sin\frac{\beta}{2}}$$

$$=\frac{4sin^2\frac{\alpha}{2}sin^2\frac{\beta}{2}+4sin\frac{\alpha}{2}sin\frac{\beta}{2}cos\frac{\alpha}{2}cos\frac{\beta}{2}cos\theta}{4sin\frac{\alpha}{2}sin\frac{\beta}{2}}$$

$$=sin\frac{\alpha}{2}sin\frac{\beta}{2}+cos\frac{\alpha}{2}cos\frac{\beta}{2}cos\theta$$

$$\boldsymbol{cos\theta_p=sin\frac{\alpha}{2}sin\frac{\beta}{2}+cos\frac{\alpha}{2}cos\frac{\beta}{2}cos\theta} \qquad \forall\ \boldsymbol{0<\alpha,\beta,\theta\le\pi}\ \Rightarrow\ \boldsymbol{0\le\theta_p<\theta}$$

The above formula is ***HCR's cosine formula*** which is dimensionless & independent of radius of the spherical surface. Hence, it is applicable on any spherical surface to find out any one of four angles α, β, θ & θ_p when other three are given (known). Above formula is extremely useful to find out the plane angle $\boldsymbol{\theta_p}$ between the chords of any two great circle arcs when $\boldsymbol{\alpha,\beta}$ & $\boldsymbol{\theta}$ are known because it is used in deriving the formula for calculating the **minimum distance between any two arbitrary points on a spherical surface of finite radius for given values of latitude & longitude of the points**.

NOTE: The value of $\boldsymbol{\theta_p}$ is always less than the value of $\boldsymbol{\theta}$ for any given values of $\boldsymbol{\alpha,\beta}$ & $\boldsymbol{\theta}$ $\boldsymbol{(0<\alpha,\beta,\theta\le\pi)}$. This is only due to the **obliquity of planes of angle $\boldsymbol{\theta_p}$ & $\boldsymbol{\theta}$** and that is also the reason why the **sum of interior angles of a spherical polygon is always greater than that of a corresponding plane polygon**. If the lengths of two great circle arcs are given then the angles $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$ subtended by them at the centre of sphere are easily calculated by the following formula

$$angle\ subtended\ by\ a\ great\ circle\ arc\ at\ the\ centre\ of\ sphere=\frac{length\ of\ great\ circle\ arc}{radius\ of\ sphere}$$

There are some important deductions from cosine formula depending on the following cases

Case 1: If two great circles arcs, subtending the angles $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$ at the centre of sphere, meet each other orthogonally then

Substituting $\theta={}^{\pi}/_{2}$ in cosine formula, the angle between chords of great circle arcs is given as

$$cos\theta_p=sin\frac{\alpha}{2}sin\frac{\beta}{2}+cos\frac{\alpha}{2}cos\frac{\beta}{2}cos\frac{\pi}{2}$$

$$\boldsymbol{cos\theta_p=sin\frac{\alpha}{2}sin\frac{\beta}{2}} \qquad \forall\ 0<\alpha,\beta\le\pi\ \Rightarrow\ 0\le\theta_p<\frac{\pi}{2}$$

The above formula is extremely useful in finding out the **minimum distance or great circle distance between any two arbitrary points on a spherical surface of finite radius**.

Case 2: If two great circles arcs, subtending equal angle $\boldsymbol{\alpha}$ at the centre of sphere, meet each other at an angle $\boldsymbol{\theta}$ then

Substituting $\beta=\alpha$ in cosine formula, the angle between chords of great circle arcs is given as

$$cos\theta_p=sin\frac{\alpha}{2}sin\frac{\alpha}{2}+cos\frac{\alpha}{2}cos\frac{\alpha}{2}cos\theta$$

$$\boldsymbol{cos\theta_p = sin^2\frac{\alpha}{2} + cos^2\frac{\alpha}{2}cos\theta} \qquad \forall\ \ 0 < \alpha, \theta \leq \pi \ \Rightarrow \ \ 0 \leq \theta_p < \theta$$

Case 3: If two great circles arcs, subtending equal angle α at the centre of sphere, meet each other orthogonally then

Substituting $\beta = \alpha$ & $\theta = {}^{\pi}/_{2}$ in cosine formula, the angle between chords of great circle arcs is given as

$$cos\theta_p = sin\frac{\alpha}{2}sin\frac{\alpha}{2} + cos\frac{\alpha}{2}cos\frac{\alpha}{2}cos\frac{\pi}{2}$$

$$\boldsymbol{cos\theta_p = sin^2\frac{\alpha}{2}} \qquad \forall\ \ 0 < \alpha \leq \pi \ \Rightarrow \ \ 0 \leq \theta_p < \frac{\pi}{2}$$

Case 4: If one of the two great circles arcs is a great semicircle $(say, \boldsymbol{\beta = \pi})$ & other subtends an angle $\boldsymbol{\alpha}$ at the centre of sphere then

Substituting $\beta = \pi$ in cosine formula, the angle between chords of great circle arcs is given as

$$cos\theta_p = sin\frac{\alpha}{2}sin\frac{\pi}{2} + cos\frac{\alpha}{2}cos\frac{\pi}{2}cos\frac{\pi}{2}$$

$$cos\theta_p = sin\frac{\alpha}{2}$$

$$cos\theta_p = cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)$$

$$\boldsymbol{\theta_p = \frac{\pi}{2} - \frac{\alpha}{2}} \qquad \forall\ \ 0 < \alpha \leq \pi \ \Rightarrow \ \ 0 \leq \theta_p < \frac{\pi}{2}$$

The above expression shows that if one of the two great circle arcs is a great semicircle (i.e. subtending an angle π at the centre of sphere) & other subtends smaller angle at the centre of sphere then the angle between the corresponding chords is independent of the angle θ between such two great circle arcs.

Derivation of formula for regular spherical polygon using HCR's cosine formula: Consider a regular spherical polygon, having $\boldsymbol{n}$ number of equal sides each as a great circle arc of length $\boldsymbol{a}$ & each interior angle $\boldsymbol{\theta}$ on a spherical surface of radius . Now, the angle subtended by each side as a great circle arc at the centre of sphere

$$= \frac{\text{Arc length of side}}{\text{radius of sphere}} = \frac{\boldsymbol{a}}{\boldsymbol{R}}$$

If we join all the vertices of the regular spherical polygon, having $\boldsymbol{n}$ number of sides each as a great circle arc, by the straight lines through the interior of sphere then we get a regular plane polygon having $\boldsymbol{n}$ number of sides then each interior angle (between any two adjacent sides) of regular plane polygon

$$= \frac{\text{total sum of interior angles}}{\text{number of sides}} = \frac{\boldsymbol{(n-2)\pi}}{\boldsymbol{n}}$$

Now, consider any two adjacent sides (as the great circle arcs of equal length subtending equal angle $\frac{a}{R}$ at the centre of sphere) meeting each other at angle θ at any vertex of the regular spherical polygon. Join the end points of these great circle arcs by the straight lines to obtain the chords meeting at the same vertex (this case will be very similar to which is shown by figure 1 above) then the (plane) angle between the chords of great circle arcs is given by HCR's cosine formula as follows

$$\cos\theta_p = \sin\frac{\alpha}{2}\sin\frac{\beta}{2} + \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\theta$$

Now, setting the corresponding values, $\theta_p = \dfrac{(n-2)\pi}{n}$,

$\alpha = \beta = \dfrac{a}{R}$ (since, the arcs are of equal length a), we get

$$\cos\frac{(n-2)\pi}{n} = \sin\frac{a}{2R}\sin\frac{a}{2R} + \cos\frac{a}{2R}\cos\frac{a}{2R}\cos\theta$$

$$\cos\left(\pi - \frac{2\pi}{n}\right) = \sin^2\frac{a}{2R} + \cos^2\frac{a}{2R}\cos\theta$$

$$-\cos\frac{2\pi}{n} = \sin^2\frac{a}{2R} + \cos^2\frac{a}{2R}\left(1 - 2\sin^2\frac{\theta}{2}\right)$$

$$-\left(2\cos^2\frac{\pi}{n} - 1\right) = \sin^2\frac{a}{2R} + \cos^2\frac{a}{2R} - 2\sin^2\frac{\theta}{2}\cos^2\frac{a}{2R}$$

$$1 - 2\cos^2\frac{\pi}{n} = 1 - 2\sin^2\frac{\theta}{2}\cos^2\frac{a}{2R}$$

$$\cos^2\frac{\pi}{n} = \sin^2\frac{\theta}{2}\cos^2\frac{a}{2R}$$

Taking square roots on both the sides, we get

$$\left|\cos\frac{\pi}{n}\right| = \left|\sin\frac{\theta}{2}\cos\frac{a}{2R}\right|$$

since, $\mathbf{n \geq 3}$, $\boldsymbol{\frac{(n-2)\pi}{n} < \theta < \pi}$ & $\boldsymbol{a < \frac{2\pi R}{n}}$ hence we get

$$\cos\frac{\pi}{n} = \sin\frac{\theta}{2}\cos\frac{a}{2R}$$

$$\boldsymbol{\sin\frac{\theta}{2}\cos\frac{a}{2R}\sec\frac{\pi}{n} = 1}$$

Above is HCR's formula for regular spherical polygon. It is very important formula which co-relates four important parameters i.e. number of sides $\boldsymbol{n}$, interior angle $\boldsymbol{\theta}$, arc length of side $\boldsymbol{a}$ & radius of sphere $\boldsymbol{R}$ in any regular spherical polygon. This formula can be used to find out any one of the four parameters if other three are known while n is always a positive integer ($n \geq 3$)

Conclusion: It's obvious that HCR's cosine formula is a dimensionless formula (angle-angle relation) which is independent of radius of the sphere & holds good for any spherical surface to find out (plane) angle between the chords of any two great arcs meeting (intersecting) each other at a common end point at some angle which is further used to find out minimum distance between any two arbitrary points on a sphere of finite radius. It can also be used to

1. Find out the angle (of concurrence) between any two great circle arcs if the angles subtended by the arcs at the centre of sphere & plane angle of the corresponding chords are known
2. Find out the angle subtended by any of two great circle arcs at the centre of sphere if angle of other great circle arc, angle of concurrence (intersection) of arcs & plane angle of the corresponding chords are known.

3. Find out the length of any of two great circle arcs if the length of other arc, radius of sphere, angle of intersection & plane angle of chords are known.
4. The interior angle of any spherical polygon if the corresponding angle of plane polygon is known. Since, the measurement of angle on a plane surface is much simpler than that on a spherical surface. Thus, this formula helps overcome the difficulty of measuring the angle of great circle arcs on a spherical surface.

Note: Above articles had been derived *& illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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