Some Important Derivations of 2D Geometry

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Introduction: Here are some **important formula** in 2D-Geometry derived by the author using simple geometry & trigonometry. The formula, derived here, are related to the triangle, square, trapezium & tangent circles. These formula are very useful for case studies in 2D-Geometry to compute the important parameters of 2D-figures. The formula & their derivations are in the order as given below.

1. If P is a point lying inside the square ABCD such that $\angle PDC = \angle PCD = \alpha$ (as shown in the figure-1) then $\angle APB = \theta$ is given by the following formula

$$\boldsymbol{\theta = 2\tan^{-1}\left(\frac{1}{2-\tan\alpha}\right)} \qquad (\forall \ \ 0 < \alpha < \tan^{-1}(2))$$

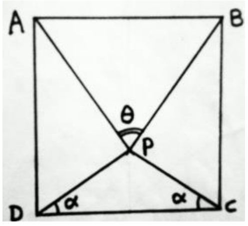

Figure 1: Given angle α

Proof: Drop the perpendiculars PM & PN from the point P to the sides AB & CD respectively in the square ABCD (as shown in the figure-2) Let $CN = DN = x$ then in right ΔPND

$$\tan\alpha = \frac{PN}{DN} = \frac{PN}{x} \Rightarrow PN = x\tan\alpha$$

$$PM = MN - PN = CD - PN = 2x - x\tan\alpha$$

In right ΔPMA

$$\tan\angle APM = {}^{AM}/_{PM}$$

$$\tan\frac{\theta}{2} = \frac{x}{2x - x\tan\alpha} = \frac{1}{2-\tan\alpha}$$

$$\theta = 2\tan^{-1}\left(\frac{1}{2-\tan\alpha}\right) \qquad \textbf{Proved.}$$

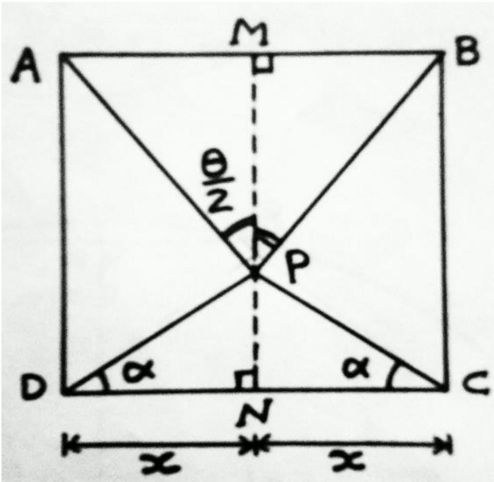

Figure 2: In square ABCD, $AM = DN = x,\ MN = CD = 2x$ & $\angle APM = \angle BPM = \theta/2$

2. If P is a point lying inside the square ABCD such that $AP = a$ & $PC = b$ (as shown in the figure-3) then the area of $\Delta\, BPD$ is given by the following formula

$$[\Delta \boldsymbol{BPD}] = \frac{|\boldsymbol{a}^2 - \boldsymbol{b}^2|}{\boldsymbol{4}}$$

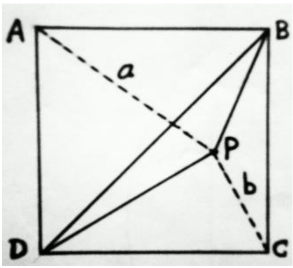

Figure 3: In square ABCD $PA = a$ & $PC = b$

Proof: Drop the perpendiculars PM & PN from the point P to the sides AD & CD respectively in the square ABCD (as shown in the figure-4 below). Let x be the side of square ABCD & $\angle PCN = \theta$ then in right ΔPNC

$$PN = b\sin\theta \ \ \& \ NC = b\cos\theta$$

$Now,$

$$MP = DN = DC - NC = x - b\cos\theta$$

$$AM = AD - MD = AD - PN = x - b\sin\theta$$

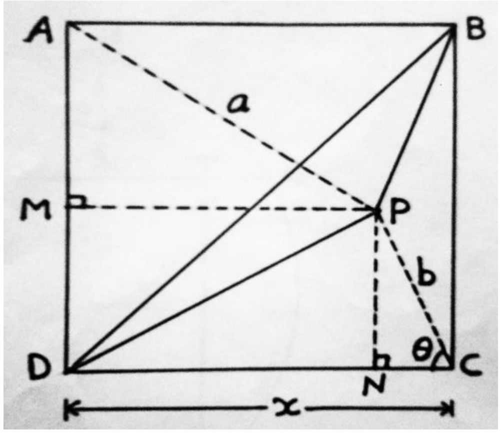

Figure 4: Let x be side of square ABCD & $\angle PCN = \theta$

Applying Pythagorean theorem in right ΔAMP

$$(AP)^2 = (AM)^2 + (MP)^2$$

$$(a)^2 = (x - b\sin\theta)^2 + (x - b\cos\theta)^2$$

$$a^2 = x^2 + b^2 sin^2\theta - 2bx\sin\theta + x^2 + b^2 cos^2\theta - 2bx\cos\theta$$

$$a^2 = 2x^2 + b^2 - 2bx(\sin\theta + \cos\theta)$$

$$2x^2 - 2bx(\sin\theta + \cos\theta) = a^2 - b^2$$

$$x^2 - bx(\sin\theta + \cos\theta) = \frac{a^2 - b^2}{2}$$

$$\frac{1}{2}\left(x^2 - bx(\sin\theta + \cos\theta)\right) = \frac{a^2 - b^2}{4} \quad \dots\dots\dots\dots\dots\dots\dots (1)$$

Now, the area of ΔBPD (refer to above figure-4) is given as

$$[\Delta BPD] = [\Delta BCD] - [\Delta BPC] - [\Delta CPD]$$

$$= \frac{1}{2}(BC)(CD) - \frac{1}{2}(BC)(NC) - \frac{1}{2}(CD)(PN) = \frac{1}{2}(x)(x) - \frac{1}{2}(x)(b\cos\theta) - \frac{1}{2}(x)(b\sin\theta)$$

$$= \frac{1}{2}(x^2 - bx\sin\theta - bx\cos\theta)$$

$$= \frac{1}{2}\left(x^2 - bx(\sin\theta + \cos\theta)\right)$$

$$= \frac{a^2 - b^2}{4} \qquad \left(\text{from eq}(1)\right)$$

Since, the area is positive hence taking absolute value of above result,

$$[\Delta BPD] = \frac{|a^2 - b^2|}{4} \qquad \textbf{Proved.}$$

3. The square ABCD has each side a. If two quarter circles each with radius a & centres at the vertices C & D are drawn then the radii R_P & R_Q of smaller inscribed circles with centres P & Q respectively (as shown in the figure-5) are given by the following formula

$$\boldsymbol{R_P = \frac{a}{16} \quad \& \quad R_Q = \frac{3a}{8}}$$

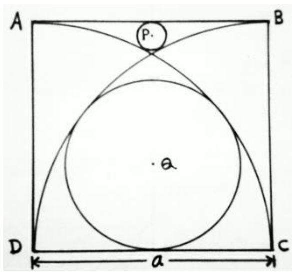

Figure 5: Two inscribed circles have radii R_P & R_Q

Proof: Let R_P & R_Q be the radii of smaller inscribed circles with the centres P & Q respectively. Join the centres P & Q to the vertex D and draw the line MN passing through the centres P & Q which is perpendicular to the sides AB & CD of the square ABCD (as shown in the figure-6).

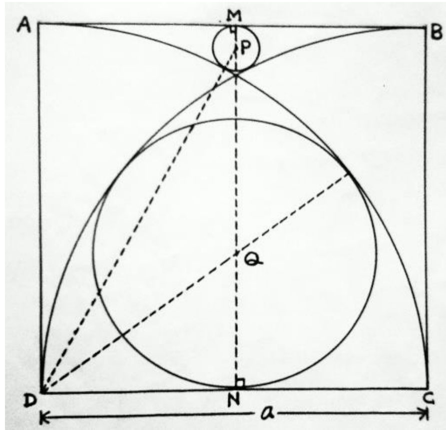

Figure 6: $PM = R_P, QN = R_Q$ $PD = a + R_P$, $PN = a - R_P$ & $DQ = a - R_Q$

Applying Pythagorean theorem in right ΔPND

$$(PD)^2 = (PN)^2 + (DN)^2$$

$$(a + R_P)^2 = (a - R_P)^2 + \left(\frac{a}{2}\right)^2$$

$$a^2 + {R_P}^2 + 2aR_P = a^2 + {R_P}^2 - 2aR_P + \frac{a^2}{4}$$

$$4aR_P = \frac{a^2}{4}$$

$$R_P = \frac{a}{16}$$ **Proved**

Now, applying Pythagorean theorem in right ΔQND

$$(QD)^2 = (QN)^2 + (DN)^2$$

$$(a - R_Q)^2 = (R_Q)^2 + \left(\frac{a}{2}\right)^2$$

$$a^2 + {R_Q}^2 - 2aR_Q = {R_Q}^2 + \frac{a^2}{4}$$

$$2aR_Q = \frac{3a^2}{4}$$

$$R_Q = \frac{3a}{8}$$ **Proved.**

4. The square ABCD has each side a. If two quarter circles each with radius a & centres at the vertices C & D are drawn then the radius (r) of smaller inscribed circle (as shown in the figure-7) is given by the following formula

$$\boldsymbol{r = \frac{a}{6}}$$

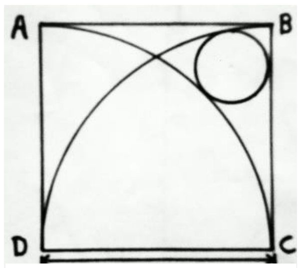

Figure 7: Small circle of radius r inscribed by two quarter circles & the side BC

Proof: Let $\boldsymbol{r}$ be the radius of smaller inscribed circle with the centre O. Join the centre O to the vertices C & D and drop the perpendiculars OM & ON from the centre O to the sides BC & CD respectively in the square ABCD (as shown in the figure-8).

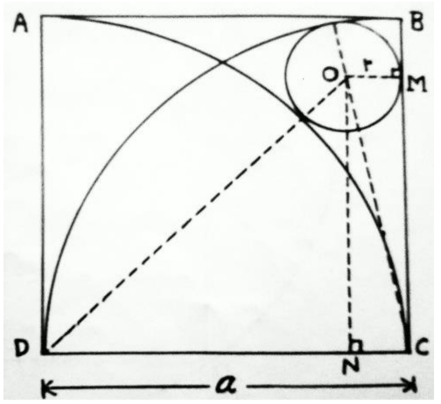

Figure 8: In square ABCD, $\boldsymbol{OC = a - r}$ & $\boldsymbol{OD = a + r}$

Applying Pythagorean theorem in right ΔOMC

$$MC = \sqrt{(OC)^2 - (OM)^2}$$

$$= \sqrt{(a-r)^2 - r^2}$$

$$= \sqrt{a^2 - 2ar + r^2 - r^2}$$

$$MC = ON = \sqrt{a^2 - 2ar}$$

Applying Pythagorean theorem in right ΔOND

$$(OD)^2 = (ON)^2 + (DN)^2$$

$$(a+r)^2 = \left(\sqrt{a^2 - 2ar}\right)^2 + (a-r)^2$$

$$a^2 + 2ar + r^2 = a^2 - 2ar + a^2 - 2ar + r^2$$

$$6ar = a^2$$

$$r = \frac{a}{6}$$ **Proved.**

5. Two squares ABCD & STUV having each side $\boldsymbol{a}$ & $\boldsymbol{b}$ respectively, are symmetrically drawn sharing a common edge coinciding with the chord PQ of a circle which inscribes the two squares (as shown in the figure-9) & passes through the vertices A, B, U & V. Then the radius ($\boldsymbol{R}$) of inscribing circle & the length of common chord PQ are given by the following formula

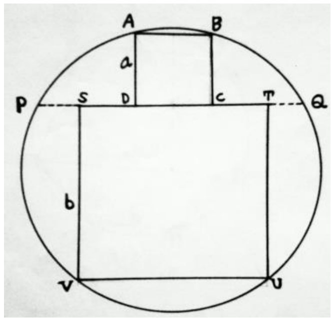

Figure 9: Two squares are drawn with a common edge inscribed in a circle

$$\boldsymbol{R = \frac{\sqrt{5(5a^2 + 5b^2 + 6ab)}}{8}} \quad \& \quad \boldsymbol{PQ = \sqrt{5ab}}$$

Proof: Let R be the radius of circle with the centre O inscribing two given squares. Join the centre O to the vertices A & V and the point P. Drop the perpendiculars OM & ON from the centre O to the sides AB & UV respectively (as shown in the figure-10).

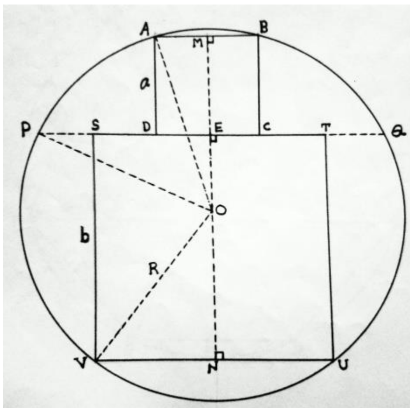

Figure 10: Two squares ABCD & STUV sharing common edge on the chord PQ

Applying Pythagorean theorem in right ΔOMA

$$OM = \sqrt{(OA)^2 - (AM)^2} = \sqrt{R^2 - \left(\frac{a}{2}\right)^2} = \frac{\sqrt{4R^2 - a^2}}{2}$$

Applying Pythagorean theorem in right ΔONV

$$ON = \sqrt{(OV)^2 - (VN)^2} = \sqrt{R^2 - \left(\frac{b}{2}\right)^2} = \frac{\sqrt{4R^2 - b^2}}{2}$$

From symmetry in the above figure 10, we have

$$OM + ON = MN$$

$$\frac{\sqrt{4R^2 - a^2}}{2} + \frac{\sqrt{4R^2 - b^2}}{2} = a + b$$

$$\sqrt{4R^2 - a^2} + \sqrt{4R^2 - b^2} = 2(a + b)$$

$$\left(\sqrt{4R^2 - a^2} + \sqrt{4R^2 - b^2}\right)^2 = \left(2(a + b)\right)^2$$

$$4R^2 - a^2 + 4R^2 - b^2 + 2\sqrt{(4R^2 - a^2)(4R^2 - b^2)} = 4(a^2 + b^2 + 2ab)$$

$$2\sqrt{(4R^2 - a^2)(4R^2 - b^2)} = 5a^2 + 5b^2 + 8ab - 8R^2$$

$$\left(2\sqrt{(4R^2 - a^2)(4R^2 - b^2)}\right)^2 = (5a^2 + 5b^2 + 8ab - 8R^2)^2$$

$$64R^4 - 16(a^2 + b^2)R^2 + 4a^2b^2 = (5a^2 + 5b^2 + 8ab)^2 + 64R^4 - 16(5a^2 + 5b^2 + 8ab)R^2$$

$$16(5a^2 + 5b^2 + 8ab - a^2 - b^2)R^2 = (5a^2 + 5b^2 + 8ab)^2 - 4a^2b^2$$

$$16(5a^2 + 5b^2 + 8ab - a^2 - b^2)R^2 = (5a^2 + 5b^2 + 8ab + 2ab)(5a^2 + 5b^2 + 8ab - 2ab)$$

$$16(4a^2 + 4b^2 + 8ab)R^2 = (5a^2 + 5b^2 + 10ab)(5a^2 + 5b^2 + 6ab)$$

$$64(a^2 + b^2 + 2ab)R^2 = 5(a^2 + b^2 + 2ab)(5a^2 + 5b^2 + 6ab)$$

$$64R^2 = 5(5a^2 + 5b^2 + 6ab)$$

$$R^2 = \frac{5(5a^2 + 5b^2 + 6ab)}{64}$$

$$\boldsymbol{R = \frac{\sqrt{5(5a^2 + 5b^2 + 6ab)}}{8}} \qquad \textbf{Proved.}$$

Now, substituting the above value of radius (R) of circle in the above expression, we get

$$OM = \sqrt{R^2 - \left(\frac{a}{2}\right)^2} = \sqrt{\left(\frac{\sqrt{5(5a^2 + 5b^2 + 6ab)}}{8}\right)^2 - \frac{a^2}{4}}$$

$$= \sqrt{\frac{25a^2 + 25b^2 + 30ab - 16a^2}{64}}$$

$$= \frac{\sqrt{9a^2 + 25b^2 + 30ab}}{8}$$

$$= \frac{\sqrt{(3a + 5b)^2}}{8}$$

$$OM = \frac{3a + 5b}{8}$$

From the above figure-10, we have

$$\boldsymbol{OE} = OM - EM = \frac{3a+5b}{8} - a = \frac{5(b-a)}{8}$$

Applying Pythagorean theorem in right ΔOEP (refer to figure -10 above)

$$PE = \sqrt{(OP)^2 - (OE)^2} = \sqrt{R^2 - (OE)^2}$$

$$= \sqrt{\left(\frac{\sqrt{5(5a^2+5b^2+6ab)}}{8}\right)^2 - \left(\frac{5(b-a)}{8}\right)^2}$$

$$= \sqrt{\frac{25a^2+25b^2+30ab-25a^2-25b^2+50ab}{64}}$$

$$= \sqrt{\frac{80ab}{64}}$$

$$PE = \frac{\sqrt{5ab}}{2}$$

Hence, from the above figure-10, the length of common chord PQ,

$$\boldsymbol{PQ} = 2PE = 2\left(\frac{\sqrt{5ab}}{2}\right) = \sqrt{\boldsymbol{5ab}} \qquad \textbf{Proved.}$$

6. Two trapeziums ABCF & CDEF each having three equal sides $\boldsymbol{a}$ & $\boldsymbol{b}$ respectively, are symmetrically drawn sharing a common edge coinciding with the chord FC of a circle which inscribes the two trapeziums (as shown in the figure-11) & passes through the vertices A, B, C, D, E & F. Then the radius ($\boldsymbol{R}$) of inscribing circle & the length of common chord FC are given by the following formula

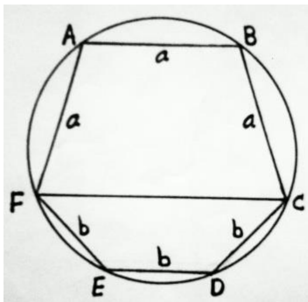

Figure 11: Two trapeziums drawn with a common edge are inscribed in a circle

$$\boldsymbol{R} = \sqrt{\frac{\boldsymbol{a^2+b^2+ab}}{\boldsymbol{3}}} \qquad \& \qquad \boldsymbol{FC} = \frac{\boldsymbol{3ab(a+b)}}{\boldsymbol{a^2+b^2+ab}}$$

Proof: Let R be the radius of circle with the centre O inscribing the two trapeziums ABCF & CDEF. Join the centre O to all the vertices A, B, C, D, E & F. Drop the perpendiculars OM & ON from the centre O to the sides AB & ED respectively (as shown in the figure-12). Let 2α & 2β be the angles exerted at the centre O by each of three equal sides a & b of trapeziums ABCF & CDEF respectively (see figure-12). Now, in right ΔOMA

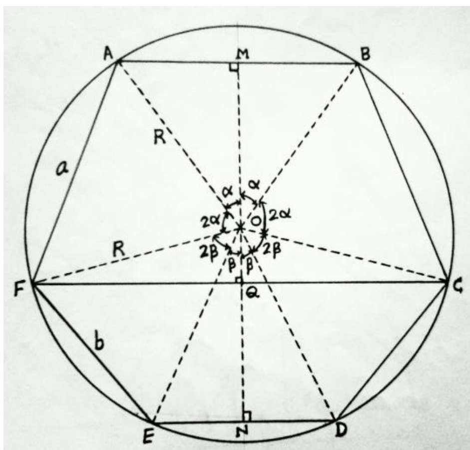

Figure 12: Two trapeziums ABCF & CDEF, each has three equal sides $\boldsymbol{a}$ & $\boldsymbol{b}$ exerting angles $2\boldsymbol{\alpha}$ & $2\boldsymbol{\beta}$ respectively at the centre O of circle inscribing them

$$\sin\alpha = \frac{AM}{OA} = \frac{a/2}{R} = \frac{a}{2R}$$

Similarly, in right ΔONE

$$\sin\beta = \frac{EN}{OE} = \frac{b/2}{R} = \frac{b}{2R}$$

Now, the sum of the angles exerted at the centre O by all the six sides of hexagon ABCDEF (see fig-12 above)

$$3(2\alpha)+3(2\beta)=360^o$$

$$\alpha+\beta=60^o$$

$$\alpha=60^o-\beta$$

$$\sin\alpha=\sin(60^o-\beta) \qquad \text{(taking sine on both sides)}$$

$$\sin\alpha=\sin 60^o\cos\beta-\cos 60^o\sin\beta$$

$$\frac{\sin\alpha}{\sin\beta}=\frac{\frac{\sqrt{3}}{2}\cos\beta-\frac{1}{2}\sin\beta}{\sin\beta} \qquad \text{(diving both sides by} \sin\beta)$$

$$\frac{\frac{a}{2R}}{\frac{b}{2R}}=\frac{\sqrt{3}}{2}\cot\beta-\frac{1}{2}$$

$$\frac{a}{b}=\frac{\sqrt{3}}{2}\cot\beta-\frac{1}{2}$$

$$\cot\beta=\frac{2a+b}{b\sqrt{3}}$$

$$Now, \qquad \sin\beta=\frac{1}{cosec\beta}=\frac{1}{\sqrt{1+cot^2\beta}} \qquad (0<\beta<\pi)$$

$$\boldsymbol{\sin\beta}=\frac{1}{\sqrt{1+\left(\frac{2a+b}{b\sqrt{3}}\right)^2}}=\frac{b\sqrt{3}}{\sqrt{4a^2+4b^2+4ab}}=\boldsymbol{\frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}}$$

But in right ΔONE (see figure -12 above)

$$\sin\beta=\frac{EN}{OE}=\frac{b/2}{R}=\frac{b}{2R}$$

$$\therefore \frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}=\frac{b}{2R}$$

$$\frac{\sqrt{3}}{\sqrt{a^2+b^2+ab}}=\frac{1}{R}$$

$$R=\sqrt{\frac{a^2+b^2+ab}{3}} \qquad \textbf{Proved}$$

Now in right ΔOQF (see figure-12 above)

$$\sin\angle FOQ=\frac{FQ}{OF}$$

$$\sin 3\beta=\frac{FQ}{R}$$

$$FQ = R\sin 3\beta$$

$$= R(3\sin\beta - 4sin^3\beta)$$

$$= R\sin\beta\,(3 - 4sin^2\beta)$$

$$= \sqrt{\frac{a^2+b^2+ab}{3}}\left(\frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}\right)\left(3-4\left(\frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}\right)^2\right)$$

$$= \frac{b}{2}\left(\frac{3a^2+3b^2+3ab-3b^2}{a^2+b^2+ab}\right)$$

$$= \frac{b}{2}\left(\frac{3a(a+b)}{a^2+b^2+ab}\right)$$

$$FQ = \frac{3ab(a+b)}{2(a^2+b^2+ab)}$$

Now, the **length of common chord FC** (see figure-12 above) is given as

$$FC = 2FQ = 2\left(\frac{3ab(a+b)}{2(a^2+b^2+ab)}\right) = \frac{3ab(a+b)}{a^2+b^2+ab}$$ **Proved.**

7. A small circle with radius r & centre at D is inscribed in a large semi-circle of radius R & centre at O such that it touches the semi-circle internally at the periphery & the diameter AB. Now, a tangent BC is drawn from end B to the small circle & extended such that it intersects semi-circle at C (As shown in the figure-13). Then the length of extended tangent BC is given by the following formula

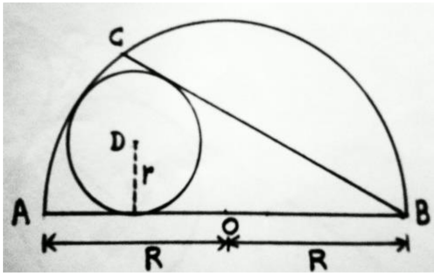

Figure 13: Small circle touches the periphery & diameter of semi-circle internally

$$\boldsymbol{BC = 2R\cos\left\{2\tan^{-1}\left(\frac{r}{R+\sqrt{R^2-2Rr}}\right)\right\} \qquad \forall\, r < R/2}$$

Proof: Join the centre D of small circle to the centre O of semi-circle & the point B by dotted straight lines. Drop the perpendiculars OM & DN from the centres O & D to the diameter AB & tangent BC respectively (as shown in the figure-14). Let 2α be the angle between diameter AB & tangent BC which is bisected by line BD (see figure-14).

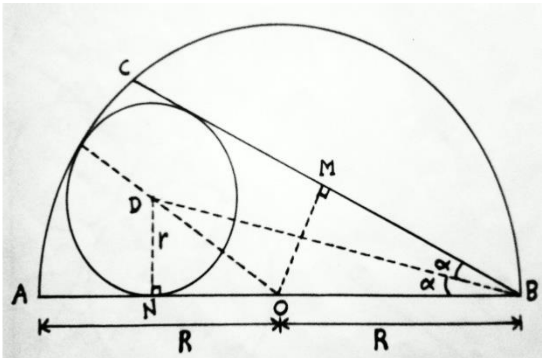

Figure 14: Perpendicular OM drawn from the centre O bisects extended tangent line BC. BD bisects $\angle ABC$

Applying Pythagorean theorem in right ΔDNO

$$NO = \sqrt{(OD)^2-(DN)^2}$$

$$= \sqrt{(R-r)^2-(r)^2}$$

$$= \sqrt{R^2+r^2-2Rr-r^2}$$

$$= \sqrt{R^2-2Rr}$$

Now, in right ΔDNB

$$\tan\alpha = \frac{DN}{BN} = \frac{r}{NO+OB} = \frac{r}{\sqrt{R^2-2Rr}+R} = \frac{r}{R+\sqrt{R^2-2Rr}}$$

$$\alpha = \tan^{-1}\left(\frac{r}{R+\sqrt{R^2-2Rr}}\right)$$

Now, in right ΔOMB (see figure 14 above)

$$\cos\angle OBM = \frac{BM}{OB}$$

$$\cos 2\alpha = \frac{BM}{R}$$

$$BM = R\cos 2\alpha$$

Hence, setting the value of the angle α, the **length of extended tangent line BC**,

$$BC = 2BM = 2R\cos 2\alpha$$

$$BC = 2R\cos\left\{2\tan^{-1}\left(\frac{r}{R+\sqrt{R^2-2Rr}}\right)\right\}$$ **Proved.**

8. Two small semi-circles of radii a & b and centres at C & D are drawn inside a large semi-circle of radius $a+b$ & centre at O such that the small semi-circles completely share the diameter AB of large one. Now two identical (twins) small circles with centres C_1 & C_2 are inscribed by three semi-circles & a perpendicular line EF such that these twins are tangent to the vertical line EF, internally tangent to big semi-circle & externally tangent to small semi-circles. (as shown in the figure-15). Then the radius (r) of each of identical (twins) circles is given by the following formula

$$\boldsymbol{r = \frac{ab}{a+b}}$$

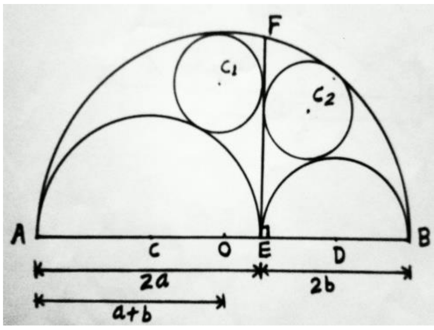

Figure 15: small identical (twins) circles with centres C_1 & C_2 are tangent to three semi-circles & perpendicular EF

Proof: Join the centres C_1 & C_2 of small identical circles to the centres C, O & O, D respectively by the dotted straight lines. Drop the perpendiculars from the centres C_1 & C_2 to the perpendicular EF & the diameter AB. (as shown in the figure-16). Let r be the radius of each of small identical circles, $\angle C_1CO = \alpha$ & $\angle C_2DO = \beta$.

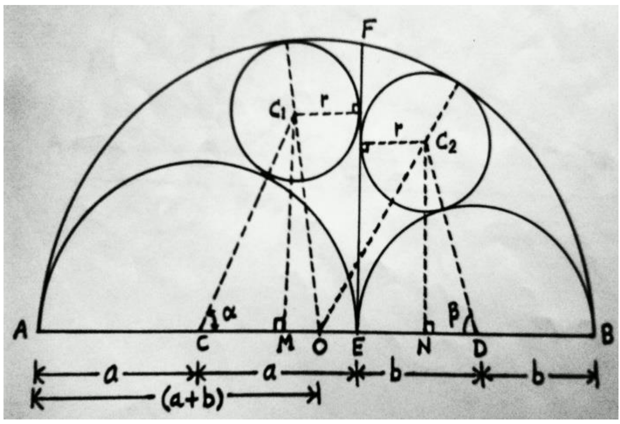

Figure 16: $OC = b$ & $CC_1 = a + r, OC_1 = a + b - r$ and $OD = a$ & $C_2D = b + r, OC_2 = a + b - r$

Now in ΔC_1CO , applying cosine rule,

$$\cos\angle C_1CO = \frac{(C_1C)^2+(CO)^2-(C_1O)^2}{2(C_1C)(CO)}$$

$$\cos\alpha = \frac{(a+r)^2+(b)^2-(a+b-r)^2}{2(a+r)(b)}$$

$$\cos\alpha = \frac{a^2+r^2+2ar+b^2-a^2-b^2-r^2-2ab+2br+2ar}{2b(a+r)}$$

$$\cos\alpha = \frac{2ar+br-ab}{b(a+r)}$$

Now, in right ΔC_1MC (see figure 16 above)

$$\cos\angle C_1CM = \frac{CM}{CC_1}$$

$$\cos\alpha = \frac{CM}{a+r}$$

$$\boldsymbol{CM} = (a+r)\cos\alpha = (a+r)\frac{2ar+br-ab}{b(a+r)} = \frac{2ar+br-ab}{b} \quad \dots\dots\dots(1)$$

Now, in ΔC_2DO , applying cosine rule,

$$\cos\angle C_2DO = \frac{(C_2D)^2+(OD)^2-(C_2O)^2}{2(C_2D)(OD)}$$

$$\cos\beta = \frac{(b+r)^2+(a)^2-(a+b-r)^2}{2(b+r)(a)}$$

$$\cos\beta = \frac{b^2+r^2+2br+a^2-a^2-b^2-r^2-2ab+2br+2ar}{2a(b+r)}$$

$$\cos\beta = \frac{2br+ar-ab}{a(b+r)}$$

Now, in right ΔC_2ND (see figure 16 above)

$$\cos\angle C_2DN = \frac{ND}{C_2D}$$

$$\cos\beta = \frac{ND}{b+r}$$

$$\boldsymbol{ND} = (b+r)\cos\beta = (b+r)\frac{2br+ar-ab}{a(b+r)} = \frac{2br+ar-ab}{a} \quad \dots\dots\dots(2)$$

Now, from the figure 16 above, we have

$$MN = CD - CM - ND$$

$$2r = a+b-\frac{2ar+br-ab}{b}-\frac{2br+ar-ab}{a} \quad \text{(Setting values of CM \& ND from eq1 \& } eq2\text{)}$$

$$2r = \frac{a^2b+ab^2-2a^2r-abr+a^2b-2b^2r-abr+ab^2}{ab}$$

$$2abr = 2(a^2b+ab^2-a^2r-b^2r-abr)$$

$$abr = a^2b+ab^2-a^2r-b^2r-abr$$

$$(a^2+b^2+2ab)r = a^2b+ab^2$$

$$(a+b)^2r = ab(a+b)$$

$$r = \frac{ab}{a+b}$$

Proved.

9. A right ΔABC with orthogonal sides a & b is divided into two small right triangles ΔBDA & ΔBDC by drawing a perpendicular BD from right-angled vertex B to the hypotenuse AC. If O_1 & O_2 are the centres of the inscribed circles of right ΔBDA & ΔBDC respectively (as shown in the figure-17) then the distance between the inscribed centres O_1 & O_2 is given by the following formula

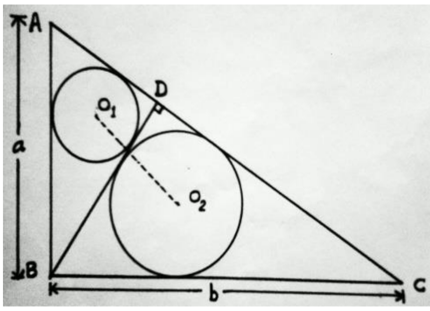

Figure 17: O_1 & O_2 are the centres of inscribed circles of right ΔBDA & ΔBDC

$$O_1O_2 = \frac{a+b-\sqrt{a^2+b^2}}{\sqrt{2}}$$

Proof: Join the centres O_1 & O_2 of inscribed circles respectively by the dotted straight lines. Drop the perpendiculars O_1P & O_2Q from the centres O_1 & O_2 respectively to the hypotenuse AC (as shown in the figure-18).

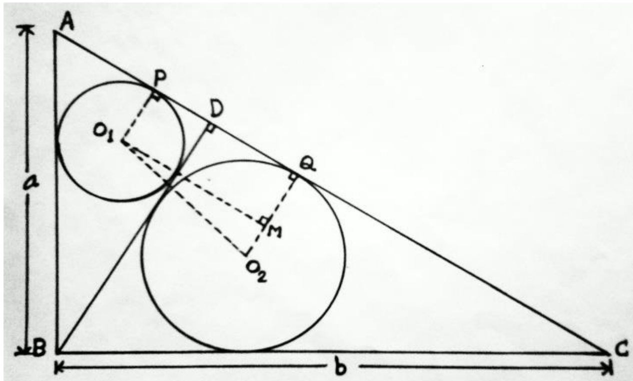

Figure 18: A perpendicular O_1M is drawn to the line O_2Q such that $O_1M = PQ$

$$Area\ of\ right\ \Delta ABC = \frac{1}{2}(AB)(BC) = \frac{1}{2}(AC)(BD)$$

$$\frac{1}{2}(a)(b) = \frac{1}{2}\left(\sqrt{a^2+b^2}\right)(BD)$$

$$BD = \frac{ab}{\sqrt{a^2+b^2}}$$

Now, applying Pythagorean theorem in right ΔBDA ,

$$\boldsymbol{AD} = \sqrt{(AB)^2-(BD)^2} = \sqrt{a^2-\left(\frac{ab}{\sqrt{a^2+b^2}}\right)^2} = \sqrt{\frac{a^4+a^2b^2-a^2b^2}{a^2+b^2}} = \frac{a^2}{\sqrt{a^2+b^2}}$$

Similarly,

$$\boldsymbol{CD} = \frac{b^2}{\sqrt{a^2+b^2}}$$

Now, the radius O_1P of inscribed circle of right ΔBDA is given as

$$O_1P = \frac{\text{Area of } \Delta\text{BDA}}{\text{semi perimeter}} = \frac{\frac{1}{2}(BD)(AD)}{\frac{1}{2}(AB+BD+AD)}$$

$$= \frac{\frac{1}{2}\left(\frac{ab}{\sqrt{a^2+b^2}}\right)\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)}{\frac{1}{2}\left(a+\frac{ab}{\sqrt{a^2+b^2}}+\frac{a^2}{\sqrt{a^2+b^2}}\right)}$$

$$= \frac{a^3b}{a^3+ab^2+(a^2+ab)\sqrt{a^2+b^2}}$$

$$= \frac{a^2b}{a^2+b^2+(a+b)\sqrt{a^2+b^2}}$$

$$=\frac{a^2b}{\sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right)}$$

$$=\frac{a^2b\left(a+b-\sqrt{a^2+b^2}\right)}{\sqrt{a^2+b^2}\left(a+b+\sqrt{a^2+b^2}\right)\left(a+b-\sqrt{a^2+b^2}\right)}$$

$$=\frac{a^2b\left(a+b-\sqrt{a^2+b^2}\right)}{\sqrt{a^2+b^2}\left((a+b)^2-\left(\sqrt{a^2+b^2}\right)^2\right)}$$

$$=\frac{a^2b\left(a+b-\sqrt{a^2+b^2}\right)}{\sqrt{a^2+b^2}(a^2+b^2+2ab-a^2-b^2)}$$

$$=\frac{a^2b\left(a+b-\sqrt{a^2+b^2}\right)}{\sqrt{a^2+b^2}(2ab)}$$

$$=\frac{a\left(a+b-\sqrt{a^2+b^2}\right)}{2\sqrt{a^2+b^2}}$$

$$\boldsymbol{O_1P=\frac{a\left(a+b-\sqrt{a^2+b^2}\right)}{2\sqrt{a^2+b^2}}} \quad \dots\dots\dots\dots\dots\dots\dots(\mathbf{1})$$

Similarly, the radius O_2Q of inscribed circle of right ΔBDC is given as

$$\boldsymbol{O_2Q=\frac{b\left(a+b-\sqrt{a^2+b^2}\right)}{2\sqrt{a^2+b^2}}} \quad \dots\dots\dots\dots\dots(\mathbf{2})$$

Now, in right ΔBDA , using **property of the inscribed circle** (refer to the figure-18 above),

$$AP=\frac{AB+AD-BD}{2}$$

$$=\frac{a+\frac{a^2}{\sqrt{a^2+b^2}}-\frac{ab}{\sqrt{a^2+b^2}}}{2}$$

$$AP=\frac{a^2-ab+a\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}} \quad \dots\dots\dots\dots\dots(3)$$

Similarly, in right ΔBDC

$$QC=\frac{b^2-ab+b\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}} \quad \dots\dots\dots\dots\dots(4)$$

Now, in right ΔABC (see figure-18 above)

$$PQ=AC-AP-QC$$

$$=\sqrt{a^2+b^2}-\frac{a^2-ab+a\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}}-\frac{b^2-ab+b\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}} \quad \left(\text{setting values from }(3)\&(4)\right)$$

$$=\frac{2a^2+2b^2-a^2+ab-a\sqrt{a^2+b^2}-b^2+ab-b\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}}$$

$$=\frac{a^2+b^2+2ab-(a+b)\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}}$$

$$=\frac{(a+b)^2-(a+b)\sqrt{a^2+b^2}}{2\sqrt{a^2+b^2}}$$

$$PQ=O_1M=\frac{(a+b)\left(a+b-\sqrt{a^2+b^2}\right)}{2\sqrt{a^2+b^2}} \quad \dots\dots\dots\dots\dots (5)$$

Applying Pythagorean theorem in right ΔO_1MO_2(see figure-18 above)

$$O_1O_2=\sqrt{(O_1M)^2+(O_2M)^2}$$

$$O_1O_2=\sqrt{(O_1M)^2+(O_2Q-O_1P)^2}$$

Setting the corresponding values of O_1P, O_2Q & O_1M from eq(1), (2) & (5) in above expression, we get

$$O_1O_2=\sqrt{\left(\frac{(a+b)\left(a+b-\sqrt{a^2+b^2}\right)}{2\sqrt{a^2+b^2}}\right)^2+\left(\frac{b\left(a+b-\sqrt{a^2+b^2}\right)}{2\sqrt{a^2+b^2}}-\frac{a\left(a+b-\sqrt{a^2+b^2}\right)}{2\sqrt{a^2+b^2}}\right)^2}$$

$$=\sqrt{\left(\frac{(a+b)\left(a+b-\sqrt{a^2+b^2}\right)}{2\sqrt{a^2+b^2}}\right)^2+\left(\frac{(b-a)\left(a+b-\sqrt{a^2+b^2}\right)}{2\sqrt{a^2+b^2}}\right)^2}$$

$$=\frac{\left(a+b-\sqrt{a^2+b^2}\right)}{2\sqrt{a^2+b^2}}\sqrt{(a+b)^2+(b-a)^2}$$

$$=\frac{\left(a+b-\sqrt{a^2+b^2}\right)}{2\sqrt{a^2+b^2}}\sqrt{a^2+b^2+2ab+a^2+b^2-2ab}$$

$$=\frac{\left(a+b-\sqrt{a^2+b^2}\right)}{2\sqrt{a^2+b^2}}\sqrt{2(a^2+b^2)}$$

$$O_1O_2=\frac{\left(a+b-\sqrt{a^2+b^2}\right)}{\sqrt{2}}$$

Proved.

10. If P is a point lying inside the ΔABC such that $\angle PAB=\angle PBC=\angle PCA=\theta$ then the angle θ is given by the following formula

$$\tan\theta=\frac{\sin A\sin B\sin C}{1+\cos A\cos B\cos C}$$

Proof: Consider any point P lying inside the ΔABC having its sides $BC=a, AC=b, AB=c$ such that $\angle PAB=\angle PBC=\angle PCA=\theta$. Join the point P to the vertices A, B & C by the dotted straight lines. Drop the perpendiculars PM & PN from the point P to the sides AB & AC respectively. (as shown in the figure-19 below).

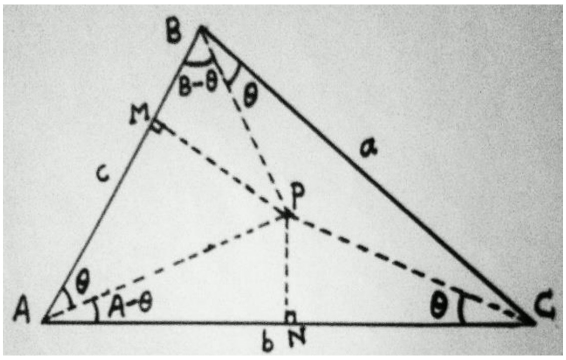

Figure 19: Point P is lying inside ΔABC having sides a, b, c such that $\angle PAB = \angle PBC = \angle PCA = \theta$

Now, in right ΔPMA,

$$\tan\angle PAM = \frac{PM}{AM} \;\Rightarrow\; \tan\theta = \frac{PM}{AM}$$

$$AM = \frac{PM}{\tan\theta}$$

Similarly, in right ΔPMB

$$MB = \frac{PM}{\tan(B-\theta)}$$

But we have, $AM + MB = AB = c$

$$\therefore \frac{PM}{\tan\theta} + \frac{PM}{\tan(B-\theta)} = c$$

$$PM = \frac{c\tan\theta\tan(B-\theta)}{\tan\theta + \tan(B-\theta)}$$

Again, in right ΔPMA,

$$\sin\theta = \frac{PM}{PA} \;\Rightarrow\; PA = \frac{PM}{\sin\theta}$$

$$= \frac{\frac{c\tan\theta\tan(B-\theta)}{\tan\theta + \tan(B-\theta)}}{\sin\theta}$$

$$PA = \frac{c\sec\theta\tan(B-\theta)}{\tan\theta + \tan(B-\theta)} \quad \ldots\ldots\ldots\ldots(1)$$

Now, in right ΔPNC (see figure 19 above),

$$\tan\angle PCN = \frac{PN}{NC} \;\Rightarrow\; \tan\theta = \frac{PN}{NC}$$

$$NC = \frac{PN}{\tan\theta}$$

Similarly, in right ΔPNA

$$AN = \frac{PN}{\tan(A-\theta)}$$

But we have, $AN + NC = AC = b$

$$\therefore \frac{PN}{\tan(A-\theta)} + \frac{PN}{\tan\theta} = b$$

$$PN = \frac{b\tan\theta\tan(A-\theta)}{\tan\theta + \tan(A-\theta)}$$

Again, in right ΔPNA (see figure 19 above)

$$\sin(A-\theta) = \frac{PN}{PA} \;\Rightarrow\; PA = \frac{PN}{\sin(A-\theta)}$$

$$=\frac{\frac{b\tan\theta\tan(A-\theta)}{\tan\theta+\tan(A-\theta)}}{\sin(A-\theta)}$$

$$PA=\frac{b\sec(A-\theta)\tan\theta}{\tan\theta+\tan(A-\theta)}\qquad\ldots\ldots\ldots\ldots(2)$$

Now, equating the values of PA from (1) & (2), we get

$$\frac{c\sec\theta\tan(B-\theta)}{\tan\theta+\tan(B-\theta)}=\frac{b\sec(A-\theta)\tan\theta}{\tan\theta+\tan(A-\theta)}$$

$$\frac{\frac{c}{\cos\theta}\frac{\sin(B-\theta)}{\cos(B-\theta)}}{\frac{\sin\theta}{\cos\theta}+\frac{\sin(B-\theta)}{\cos(B-\theta)}}=\frac{\frac{b}{\cos(A-\theta)}\frac{\sin\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}+\frac{\sin(A-\theta)}{\cos(A-\theta)}}$$

$$\frac{c\sin(B-\theta)}{\sin\theta\cos(B-\theta)+\cos\theta\sin(B-\theta)}=\frac{c\sin\theta}{\sin\theta\cos(A-\theta)+\cos\theta\sin(A-\theta)}$$

$$\frac{c\sin(B-\theta)}{\sin(\theta+B-\theta)}=\frac{b\sin\theta}{\sin(\theta+A-\theta)}$$

$$\frac{c\sin(B-\theta)}{\sin B}=\frac{b\sin\theta}{\sin A}$$

$$\frac{\sin(B-\theta)}{\sin\theta}=\frac{b\sin B}{c\sin A}$$

$$\frac{\sin B\cos\theta-\cos B\sin\theta}{\sin\theta}=\left(\frac{b}{c}\right)\frac{\sin B}{\sin A}$$

$$\sin B\cot\theta-\cos B=\left(\frac{\sin B}{\sin C}\right)\frac{\sin B}{\sin A}\qquad\left(\text{from sine rule,}\quad\frac{b}{c}=\frac{\sin B}{\sin C}\right)$$

$$\sin B\cot\theta=\frac{sin^2 B}{\sin A\sin C}+\cos B$$

$$\cot\theta=\frac{\sin B}{\sin A\sin C}+\frac{\cos B}{\sin B}$$

$$=\frac{sin^2 B+\sin A\cos B\sin C}{\sin A\sin B\sin C}$$

$$=\frac{1-cos^2 B+\sin A\cos B\sin C}{\sin A\sin B\sin C}$$

$$=\frac{1+\cos B\,(\sin A\sin C-\cos B)}{\sin A\sin B\sin C}$$

$$=\frac{1+\cos B\,(\sin A\sin C-\cos(\pi-A-C))}{\sin A\sin B\sin C}$$

$$=\frac{1+\cos B\,(\sin A\sin C+\cos(A+C))}{\sin A\sin B\sin C}$$

$$=\frac{1+\cos B\,(\sin A\sin C+\cos A\cos C-\sin A\sin C)}{\sin A\sin B\sin C}$$

$$\cot\theta = \frac{1+\cos B\,(\cos A\cos C)}{\sin A\sin B\sin C}$$

$$\cot\theta = \frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}$$

$$\tan\theta = \frac{\sin A\sin B\sin C}{1+\cos A\cos B\cos C} \qquad \textbf{Proved.}$$

11. If the rhombus MBND is formed by joining the vertices B & D to the mid-points P, Q, R & S of the sides each of length a of a square ABCD (as shown in the figure-20) then each side, acute interior angle & the area of rhombus $MBND$ are given by the following formula

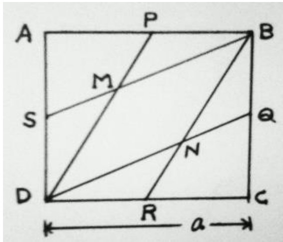

Figure 20: P, Q, R & S are the mid-points of sides of square ABCD. MBND is a rhombus

$$MB = BN = ND = DM = \frac{a\sqrt{5}}{3}, \quad \angle MDN = 2\tan^{-1}\left(\frac{1}{3}\right) \ \ \&\ \ \text{Area of rhombus } MBND = \frac{a^2}{3}$$

Proof: Draw the diagonals AC & BD of square ABCD by dotted straight lines which intersect each other at right angle at the point O. (as shown in the figure-21). Let $\angle OBN = \angle ODN = \theta$

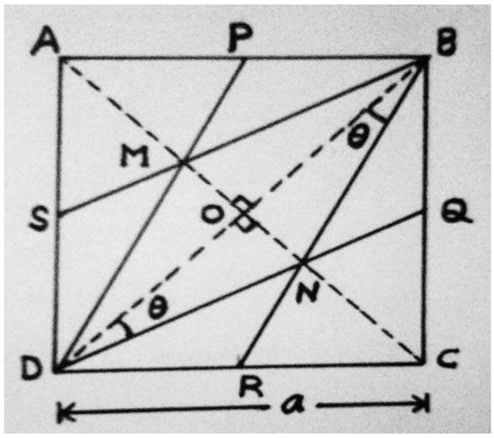

Figure 21: Rhombus MBND is obtained by joining the vertex B to the mid-points R & S & vertex D to the mid-points P & Q of sides of square ABCD

Using Pythagorean theorem in right ΔQCD

$$DQ = \sqrt{(QC)^2+(CD)^2} = \sqrt{\left(\frac{a}{2}\right)^2+(a)^2} = \frac{a\sqrt{5}}{2}$$

Now, in ΔBDQ , applying cosine rule,

$$\cos\angle BDQ = \frac{(BD)^2+(DQ)^2-(BQ)^2}{2(BD)(DQ)}$$

$$\cos\theta = \frac{\left(a\sqrt{2}\right)^2+\left(\frac{a\sqrt{5}}{2}\right)^2-\left(\frac{a}{2}\right)^2}{2\left(a\sqrt{2}\right)\left(\frac{a\sqrt{5}}{2}\right)}$$

$$= \frac{3a^2}{a^2\sqrt{10}}$$

$$= \frac{3}{\sqrt{10}}$$

$$\therefore\ \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-cos^2\theta}}{\cos\theta} \qquad \left(\text{since, } \theta < \frac{\pi}{2}\right)$$

$$\tan\theta = \frac{\sqrt{1-\left(\frac{3}{\sqrt{10}}\right)^2}}{\frac{3}{\sqrt{10}}} \quad \left(\text{setting value of } \cos\theta = {}^{3}/_{\sqrt{10}}\right)$$

$$\tan\theta = \frac{1}{3}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{1}{3}\right)$$

Hence, **acute interior angle of rhombus MBND** is given as

$$\angle MDN = \angle MBN = 2\theta = 2\tan^{-1}\left(\frac{1}{3}\right) \qquad \textbf{Proved}$$

In right ΔDON (see above figure-21)

$$\cos\theta = \frac{OD}{DN}$$

$$\Rightarrow DN = \frac{OD}{\cos\theta} = \frac{\frac{a\sqrt{2}}{2}}{\frac{3}{\sqrt{10}}} = \frac{a\sqrt{5}}{3}$$

Hence, **each side of rhombus MBND**

$$MB = BN = ND = DM = \frac{a\sqrt{5}}{3} \qquad \textbf{Proved.}$$

Again, in right ΔDON (see above figure-21)

$$\tan\theta = \frac{ON}{OD}$$

$$\Rightarrow \boldsymbol{ON} = OD\tan\theta = \left(\frac{a\sqrt{2}}{2}\right)\frac{1}{3} = \boldsymbol{\frac{a}{3\sqrt{2}}}$$

Now, using symmetry in above figure-21 , the area of rhombus MBND is given as

$$\textbf{Area of rhombus MBND} = 2(\text{Area of } \Delta\text{BDN})$$

$$= 2\left(\frac{1}{2}(BD)(ON)\right)$$

$$= 2\left(\frac{1}{2}(a\sqrt{2})\left(\frac{a}{3\sqrt{2}}\right)\right)$$

$$= \frac{a^2}{3} \qquad \textbf{Proved.}$$

$$\therefore \textbf{Area of rhombus MBND} = \frac{\mathbf{1}}{\mathbf{3}}(\textbf{Area of square ABCD})$$

12. The square ABCD has each side a. If two quarter circles each with radius a & centres at the vertices C & D and a semicircle with diameter AB are drawn then the radius (r) of smaller inscribed circle with centre O (as shown in the figure-22) is given by the following formula

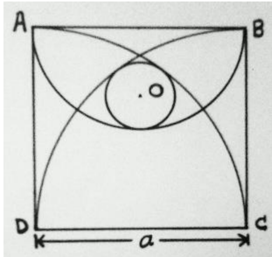

Figure 22: Small circle of radius r inscribed by two quarter circles & a semicircle

$$r = \frac{a}{6}$$

Proof: Let r be the radius of smaller inscribed circle with the centre O. Join the centre O to the vertex D & extend it to intersect quarter circle at point E. Drop the perpendicular OM from the centre O to the side CD in the square ABCD (as shown in the figure-23).

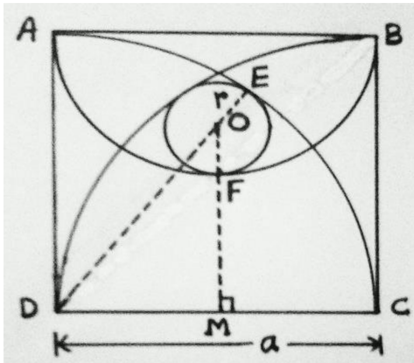

Figure-23: $DE = CD = a, OE = OF = r$ & $OD = DE - OE = a - r$

Now, in right ΔOMD, we have

$$OM = OF + FM = r + \frac{a}{2} = \frac{a}{2} + r$$

$$OD = DE - OE = a - r, \qquad DM = \frac{CD}{2} = \frac{a}{2}$$

Applying Pythagorean theorem in right ΔOMD

$$(OD)^2 = (OM)^2 + (DM)^2$$

$$(a - r)^2 = \left(\frac{a}{2} + r\right)^2 + \left(\frac{a}{2}\right)^2$$

$$a^2 + r^2 - 2ar = \frac{a^2}{4} + r^2 + ar + \frac{a^2}{4}$$

$$3ar = \frac{a^2}{2}$$

$$r = \frac{a^2}{6a}$$

$$r = \frac{a}{6}$$ **Proved.**

13. Two circles of radii r_1 & r_2 and centres A & B respectively intersect each other at two distinct points C & D such that the distance between their centres is d then the length (l) of common chord CD & angle (θ) of intersection of circles are given by following formula

$$l = \frac{\sqrt{((r_1 + r_2)^2 - d^2)(d^2 - (r_1 - r_2)^2)}}{d} \ \& \ \theta = \sin^{-1}\left(\frac{l}{2r_1}\right) + \sin^{-1}\left(\frac{l}{2r_2}\right)$$

$$Where, \qquad |r_1 - r_2| < d < (r_1 + r_2)$$

Proof: Consider two intersecting circles with centres A & B and radii r_1 & r_2 respectively. Join the centres A & B to each other & to the point of intersection C. The straight line AB bisects the common chord CD perpendicularly. Draw two tangents CP & CQ at the point of intersection C which when extended meet line AB at the points P & Q. (as shown in the figure-24).

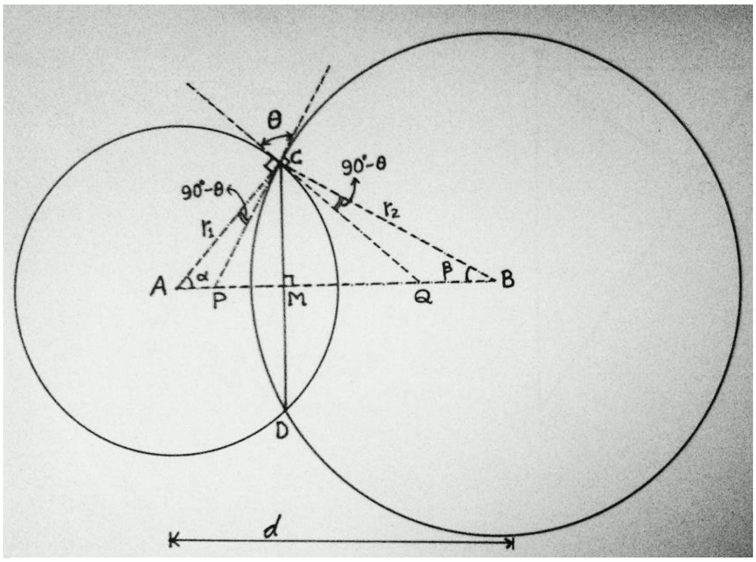

Figure-24: The common chord CD is bisected by line AB perpendicularly & angle of intersection of circles is $\angle PCQ = \theta$ between tangents CP & CQ. $\angle ACP = \angle BCQ = 90^o - \theta$

Let $CD = l, \angle PCQ = \theta$

Now, using Pythagorean theorem in right ΔAMC

$$AM = \sqrt{(AC)^2 - (CM)^2} = \sqrt{{r_1}^2 - \left(\frac{l}{2}\right)^2} = \sqrt{{r_1}^2 - \frac{l^2}{4}}$$

Now, using Pythagorean theorem in right ΔBMC

$$MB = \sqrt{(BC)^2 - (CM)^2} = \sqrt{{r_2}^2 - \left(\frac{l}{2}\right)^2} = \sqrt{{r_2}^2 - \frac{l^2}{4}}$$

But $AM + MB = AB = d$

$$\therefore \sqrt{{r_1}^2 - \frac{l^2}{4}} + \sqrt{{r_2}^2 - \frac{l^2}{4}} = d$$

$$\left(\sqrt{{r_1}^2 - \frac{l^2}{4}} + \sqrt{{r_2}^2 - \frac{l^2}{4}}\right)^2 = d^2$$

$${r_1}^2 - \frac{l^2}{4} + {r_2}^2 - \frac{l^2}{4} + 2\sqrt{\left({r_1}^2 - \frac{l^2}{4}\right)\left({r_2}^2 - \frac{l^2}{4}\right)} = d^2$$

$$d^2 - {r_1}^2 - {r_2}^2 + \frac{l^2}{2} = 2\sqrt{\left({r_1}^2 - \frac{l^2}{4}\right)\left({r_2}^2 - \frac{l^2}{4}\right)}$$

$$\left(d^2 - {r_1}^2 - {r_2}^2 + \frac{l^2}{2}\right)^2 = \left(2\sqrt{\left({r_1}^2 - \frac{l^2}{4}\right)\left({r_2}^2 - \frac{l^2}{4}\right)}\right)^2$$

$$(d^2 - {r_1}^2 - {r_2}^2)^2 + \left(\frac{l^2}{2}\right)^2 + 2\left(\frac{l^2}{2}\right)(d^2 - {r_1}^2 - {r_2}^2) = 4\left({r_1}^2 - \frac{l^2}{4}\right)\left({r_2}^2 - \frac{l^2}{4}\right)$$

$$(d^2 - {r_1}^2 - {r_2}^2)^2 + \frac{l^4}{4} + l^2(d^2 - {r_1}^2 - {r_2}^2) = 4{r_1}^2{r_2}^2 - l^2({r_1}^2 + {r_2}^2) + \frac{l^4}{4}$$

$$l^2(d^2 - {r_1}^2 - {r_2}^2 + {r_1}^2 + {r_2}^2) = 4{r_1}^2{r_2}^2 - (d^2 - {r_1}^2 - {r_2}^2)^2$$

$$l^2d^2 = (2r_1r_2 - d^2 + {r_1}^2 + {r_2}^2)(2r_1r_2 + d^2 - {r_1}^2 - {r_2}^2)$$

$$l^2 = \frac{({r_1}^2 + {r_2}^2 + 2r_1r_2 - d^2)\left(d^2 - ({r_1}^2 + {r_2}^2 - 2r_1r_2)\right)}{d^2}$$

$$l^2 = \frac{((r_1 + r_2)^2 - d^2)(d^2 - (r_1 - r_2)^2)}{d^2}$$

$$l = \frac{\sqrt{((r_1 + r_2)^2 - d^2)(d^2 - (r_1 - r_2)^2)}}{d}$$ **Proved.**

Now, in right ΔAMC, (see above figure 24) we have

$$\sin \angle CAM = \frac{CM}{AC}$$

$$\sin \alpha = \frac{l/2}{r_1} = \frac{l}{2r_1} \quad \Rightarrow \boldsymbol{\alpha = \sin^{-1}\left(\frac{l}{2r_1}\right)}$$

Similarly, in right ΔBMC, we have

$$\sin \angle CBM = \frac{CM}{BC}$$

$$\sin \beta = \frac{l/2}{r_2} = \frac{l}{2r_2} \quad \Rightarrow \boldsymbol{\beta = \sin^{-1}\left(\frac{l}{2r_2}\right)}$$

In right ΔACQ, we have $\angle ACP + \angle PCQ = \angle ACQ$

$$\angle ACP + \theta = 90^o$$

$$\angle \boldsymbol{ACP} = 90^o - \theta$$

In right ΔBCP, we have $\angle BCQ + \angle PCQ = \angle BCP$

$$\angle BCQ + \theta = 90^o$$

$$\angle \boldsymbol{BCQ} = 90^o - \theta$$

$\angle CPQ$ is an exterior angle of ΔACP, we have $\angle CPQ = \angle CAP + \angle ACP$

$$\angle \boldsymbol{CPQ} = \alpha + 90^o - \theta = \sin^{-1}\left(\frac{l}{2r_1}\right) + 90^o - \theta$$

Similarly, $\angle CQP$ is an exterior angle of ΔBCQ, we have $\angle CQP = \angle CBQ + \angle BCQ$

$$\angle \boldsymbol{CQP} = \beta + 90^o - \theta = \sin^{-1}\left(\frac{l}{2r_2}\right) + 90^o - \theta$$

Now, in ΔCPQ the sum of all interior angles is 180^o hence we have

$$\angle PCQ + \angle CPQ + \angle CQP = 180^o$$

$$\theta + \left(\sin^{-1}\left(\frac{l}{2r_1}\right) + 90^o - \theta\right) + \left(\sin^{-1}\left(\frac{l}{2r_2}\right) + 90^o - \theta\right) = 180^o$$

$$\theta = \sin^{-1}\left(\frac{l}{2r_1}\right) + \sin^{-1}\left(\frac{l}{2r_2}\right)$$ **Proved.**

14. In right ΔABC, a perpendicular BD is drawn from the right angled vertex B to the hypotenuse AC (as shown in the figure-25) then the length of perpendicular BD is given by the following formula

$$\boldsymbol{BD = \frac{AB \times BC}{AC} = \sqrt{AD \times DC}}$$

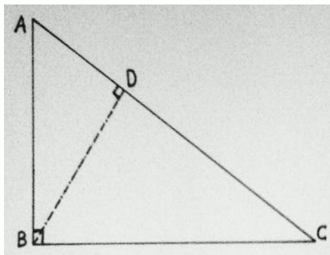

Figure 25: A perpendicular BD is drawn from right angled vertex B to the hypotenuse AC

Proof: Let $\angle BAC = A$ & $\angle ACB = C$ be two acute angles of right ΔABC. Drop a perpendicular BD from right angled vertex B to the hypotenuse AC (as shown in the figure-26).

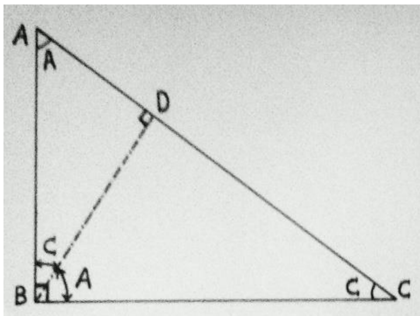

Figure-26: The perpendicular BD divides right ΔABC into two similar right triangles ΔADB & ΔBDC

Now, the area of right ΔABC is given as follows

$$\text{Area of } \Delta\text{ABC} = \frac{1}{2}AB \times BC = \frac{1}{2}AC \times BD$$

$$\frac{1}{2}AB \times BC = \frac{1}{2}AC \times BD$$

$$AB \times BC = AC \times BD$$

$$BD = \frac{AB \times BC}{AC} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Now, in right triangles ΔADB & ΔBDC, we have

$$\angle BAD = \angle CBD = A, \quad \angle ABD = \angle BCD = C, \quad \angle BDA = \angle BDC = 90^o$$

Now, from Angle-angle-angle (A-A-A) similarity, right triangles ΔADB & ΔBDC are similar triangles

Taking the ratio of corresponding sides in similar right triangles ΔADB & ΔBDC (see above figure-26)

$$\frac{BD}{DC} = \frac{AD}{BD}$$

$$(BD)^2 = AD \times DC$$

$$BD = \sqrt{AD \times DC} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

From (1) & (2), we have

$$BD = \frac{AB \times BC}{AC} = \sqrt{AD \times DC} \qquad \textbf{Proved.}$$

15. In ΔABC, three straight lines are drawn from the vertices A, B & C at an equal angle θ which intersect each other at the point O. (as shown in the figure-27) then the angle θ is given by the following formula

$$\boldsymbol{\cot\theta = \cot A + \cot B + \cot C}$$

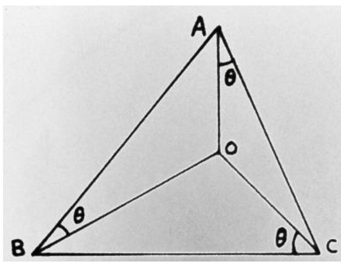

Figure 27: Three straight lines are drawn from vertices at an equal angle θ which intersect at point O

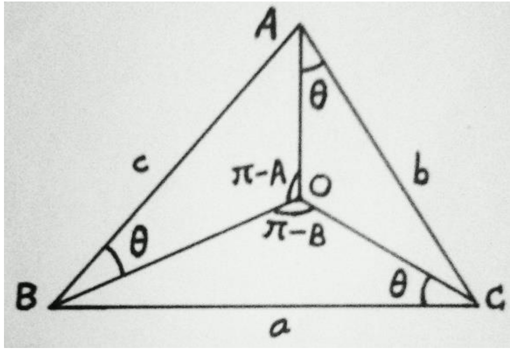

Figure-28: Three straight lines drawn at an equal angle θ from vertices of ΔABC meet at the point O

Proof: Let $AB = c, BC = a$ & $AC = b$ in ΔABC. (as shown in the figure-28).

Now, applying Sine rule in ΔABC as follows

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \Rightarrow \frac{a}{c} = \frac{\sin A}{\sin C}$$

In ΔAOB, we have

$$\angle AOB = \pi - \angle ABO - \angle BAO = \pi - \theta - (A - \theta) = \pi - A$$

Now, applying Sine rule in In ΔAOB

$$\frac{OB}{\sin(A-\theta)} = \frac{AB}{\sin(\pi - A)}$$

$$\frac{OB}{\sin(A-\theta)} = \frac{c}{\sin A} \qquad \dots\dots\dots\dots..(1)$$

In ΔBOC, we have

$$\angle BOC = \pi - \angle OBC - \angle OCB = \pi - (B - \theta) - \theta = \pi - B$$

Now, applying Sine rule in In ΔBOC

$$\frac{OB}{\sin\theta} = \frac{BC}{\sin(\pi - B)}$$

$$\frac{OB}{\sin\theta} = \frac{a}{\sin B} \qquad \dots\dots\dots\dots..(2)$$

Now, dividing (2) by (1) as follows

$$\frac{\frac{OB}{\sin\theta}}{\frac{OB}{\sin(A-\theta)}} = \frac{\frac{a}{\sin B}}{\frac{c}{\sin A}}$$

$$\frac{\sin(A-\theta)}{\sin\theta} = \left(\frac{a}{c}\right)\frac{\sin A}{\sin B}$$

$$\frac{\sin A\cos\theta - \cos A\sin\theta}{\sin\theta} = \left(\frac{\sin A}{\sin C}\right)\frac{\sin A}{\sin B}$$

$$\sin A\cot\theta - \cos A = \frac{\sin^2 A}{\sin B\sin C}$$

$$\frac{\sin A\cot\theta - \cos A}{\sin A} = \frac{\frac{\sin^2 A}{\sin B\sin C}}{\sin A} \qquad (\text{diving by} \sin A \ \ on\ both\ sides)$$

$$\cot\theta - \cot A = \frac{\sin A}{\sin B\sin C}$$

$$\cot\theta = \cot A + \frac{\sin(\pi - B - C)}{\sin B\sin C} \qquad (\text{since A + B + C} = \pi)$$

$$\cot\theta = \cot A + \frac{\sin(B + C)}{\sin B\sin C}$$

$$\cot\theta = \cot A + \frac{\sin B\cos C + \cos B\sin C}{\sin B\sin C}$$

$$\cot\theta = \cot A + \frac{\sin B\cos C}{\sin B\sin C} + \frac{\cos B\sin C}{\sin B\sin C}$$

$$\cot\theta = \cot A + \cot C + \cot B$$

$$\cot\theta = \cot A + \cot B + \cot C \qquad \textbf{Proved.}$$

16. A ΔABC is circumscribed by a circle & three altitudes AD, BE & CF are drawn from the vertices A, B & C respectively which intersect each other at the point P (orthocentre). Now, a straight line, drawn from the orthocentre P passing through the mid-point H of side BC, when extended intersects the circumscribed circle at the point G (as shown in the figure-29) then prove that straight line AG is always the diameter of circumscribed circle of ΔABC

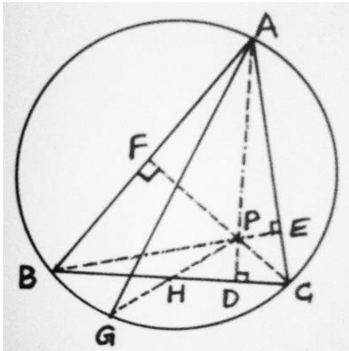

Figure 29: Straight line PH when extended intersects the circumscribed circle at point G ($BH = HC$)

Proof: Consider any ΔABC with the vertices $A(a, b)$, $B(0, 0)$ & $C(c, 0)$. Draw the altitudes AD & BE from the vertices A & B respectively to get orthocentre P. Draw a straight line from orthocentre P passing through the mid-point H of side BC which when extended intersects the circumscribed circle at the point G. (as shown in the figure-30). Now, let the centre $M(x_1, y_1)$ & radius r of circumscribed circle hence its equation is given as follows

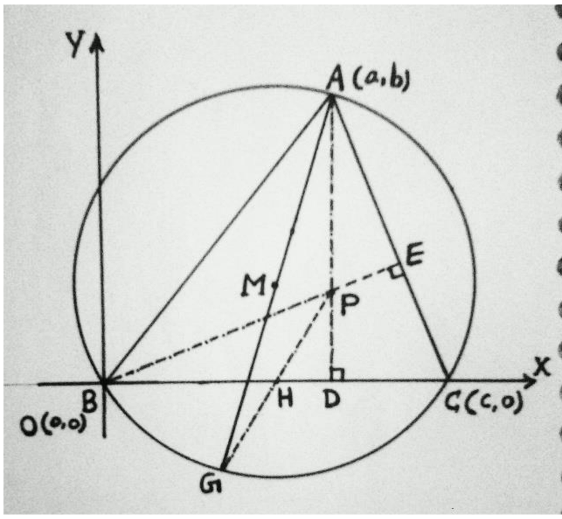

Figure 30: Straight line PH when extended intersects the circumscribed circle at point G ($BH = HC$)

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

Above circumscribed circle passes through the vertices $A(a, b)$, $B(0, 0)$ & $C(c, 0)$ hence satisfying the above equation by coordinates of the vertices as follows

$$(a - x_1)^2 + (b - y_1)^2 = r^2 \qquad \dots\dots\dots\dots(1)$$

$$(0 - x_1)^2 + (0 - y_1)^2 = r^2$$

$$x_1^2 + y_1^2 = r^2 \qquad \dots\dots\dots\dots(2)$$

$$(c - x_1)^2 + (0 - y_1)^2 = r^2$$

$$(c - x_1)^2 + y_1^2 = r^2 \qquad \dots\dots\dots\dots\dots(3)$$

Substracting (3) from (2) we get

$$x_1^2 + y_1^2 - (c - x_1)^2 - y_1^2 = r^2 - r^2$$

$$2cx_1 - c^2 = 0 \Rightarrow \ x_1 = \frac{c}{2}$$

Substracting (1) from (2) we get

$$x_1{}^2 + y_1{}^2 - (a - x_1)^2 - (b - y_1)^2 = r^2 - r^2$$

$$-a^2 + 2ax_1 - b^2 + 2by_1 = 0$$

$$y_1 = \frac{a^2 + b^2 - 2ax_1}{2b}$$

$$y_1 = \frac{a^2 + b^2 - 2a\left(\frac{c}{2}\right)}{2b} = \frac{a^2 + b^2 - ac}{2b}$$

$$\therefore M \equiv (x_1, y_1) \equiv \left(\frac{c}{2}, \frac{a^2 + b^2 - ac}{2b}\right)$$

Equation of altitude AD: $x = a$ (since, $\mathrm{BD} = \mathrm{a}$) (see above figure-30)

Equation of altitude BE passing through the origin $O(0,0)$ & perpendicular to the side AC by using slope-point form of straight line

$$y - 0 = \frac{-1}{\frac{b-0}{a-c}}(x - 0)$$

$$y = \frac{(c-a)}{b}x$$

Setting $x = a$ in above equation, we get the coordinates of orthocentre P of ΔABC as follows

$$y = \frac{(c-a)}{b}a = \frac{a(c-a)}{b}$$

$$\therefore P \equiv \left(a, \frac{a(c-a)}{b}\right)$$

Now, the equation straight line PH passing through the points $P\left(a, \frac{a(c-a)}{b}\right)$ & $H\left(\frac{c}{2}, 0\right)$ is given as folows

$$y - 0 = \frac{\frac{a(c-a)}{b} - 0}{a - \frac{c}{2}}\left(x - \frac{c}{2}\right)$$

$$y = \frac{a(c-a)}{b(2a-c)}(2x - c)$$

Now, **assume that AG is the diameter of circumscribed circle.** Since the centre $M\left(\frac{c}{2}, \frac{a^2+b^2-ac}{2b}\right)$ is the mid-point of the diameter AG passing through end-points $A(a,b)$ & G hence the coordinates of end point G are given from mid-point formula

$$G \equiv \left(2\left(\frac{c}{2}\right) - a, 2\left(\frac{a^2 + b^2 - ac}{2b}\right) - b\right) \equiv \left(c - a, \frac{a(a-c)}{b}\right)$$

Now, substituting the coordinates of point $G\left(c-a, \frac{a(a-c)}{b}\right)$ into the equation of line PH: $y=\frac{a(c-a)}{b(2a-c)}(2x-c)$ as follows

$$\frac{a(a-c)}{b}=\frac{a(c-a)}{b(2a-c)}(2(c-a)-c)$$

$$\frac{a(a-c)}{b}=\frac{a(c-a)}{b(2a-c)}(c-2a)$$

$$\frac{a(a-c)}{b}=\frac{a(a-c)}{b}$$

$$LHS=RHS$$

Above result shows that the point G satisfies the equaion of line PH i.e. the point G lies on the line PH when we assume the line AG to be the diameter of circumscribed circle. Therefore our assumption that AG is diameter of circumscribed circle is correct. Hence, the line AG is always the diameter of circumscribed circle. **Proved.**

17. A semi-circle of radius r & centre O is inscribed by a right ΔABC such that its diameter coincides with the leg BC & it touches hypotenuse AC (as shown in the fig-31). If the length of short leg AB is a then the length of other leg BC (>AB) is given by the following formula

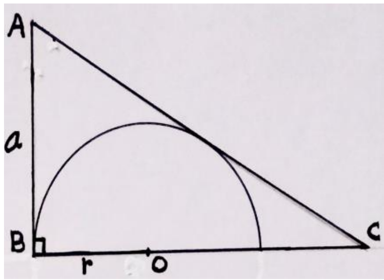

Fig-31: Semi-circle of radius r and centre O at BC, touches hypotenuse AC

$$BC=\frac{2a^2r}{a^2-r^2} \qquad (\forall\ a>r>0)$$

Proof: Consider a right ΔABC with short leg $AB=a$ inscribing a semicircle of radius r & centre O lying at leg BC such that semicircle touches the hypotenuse AC at point E. Join the point E to the centre O by a dotted straight line (As shown in the fig-32). Let $CD=x$

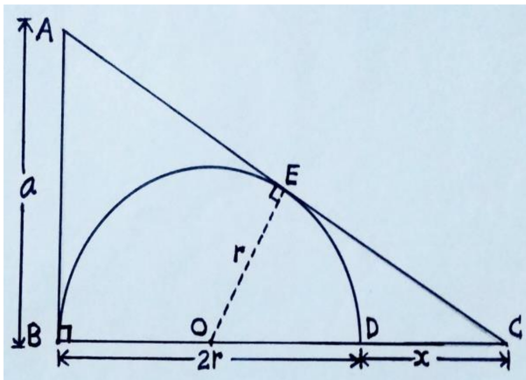

Fig-32: A semi-circle with radius r touches the hypotenuse AC at point E such that $AE=AB=a$

Using Pythagoras theorem in right ΔOEC,

$$OE^2+CE^2=OC^2$$

$$r^2+CE^2=(r+x)^2$$

$$CE=\sqrt{(r+x)^2-r^2}$$

$$CE=\sqrt{x^2+2rx} \qquad \dots\dots\dots\dots\dots(1)$$

Since, the straight lines AB & AE are two tangents to the semi-circle from the external point A hence these are equal in length i.e. $AE=AB=a$ hence in right ΔABC, we have

$$AC=AE+CE$$

$$AC=a+\sqrt{x^2+2rx} \qquad (\text{setting value of CE from (1)})$$

Using Pythagoras theorem in right ΔABC,

$$AB^2 + BC^2 = AC^2$$

$$a^2 + (2r + x)^2 = \left(a + \sqrt{x^2 + 2rx}\right)^2$$

$$a^2 + 4r^2 + x^2 + 4rx = a^2 + x^2 + 2rx + 2a\sqrt{x^2 + 2rx}$$

$$4r^2 + 2rx = 2a\sqrt{x^2 + 2rx}$$

$$(2r^2 + rx)^2 = \left(a\sqrt{x^2 + 2rx}\right)^2$$

$$4r^4 + r^2x^2 + 4r^3x = a^2x^2 + 2a^2rx$$

$$(a^2 - r^2)x^2 + 2r(a^2 - 2r^2)x - 4r^4 = 0$$

Solving above quadratic equation for x as follows

$$x = \frac{-2r(a^2 - 2r^2) \pm \sqrt{\left(2r(a^2 - 2r^2)\right)^2 - 4(a^2 - r^2)(-4r^4)}}{2(a^2 - r^2)}$$

$$= \frac{-2r(a^2 - 2r^2) \pm 2r\sqrt{(a^2 - 2r^2)^2 + 4r^2(a^2 - r^2)}}{2(a^2 - r^2)}$$

$$= \frac{2r^3 - a^2r \pm r\sqrt{a^4 + 4r^4 - 4a^2r^2 + 4a^2r^2 - 4r^4}}{a^2 - r^2}$$

$$= \frac{2r^3 - a^2r \pm r\sqrt{a^4}}{a^2 - r^2}$$

$$= \frac{2r^3 - a^2r \pm a^2r}{a^2 - r^2}$$

Case 1: Taking positive sign, we get

$$x = \frac{2r^3 - a^2r + a^2r}{a^2 - r^2} = \frac{2r^3}{a^2 - r^2}$$

Case 1: Taking negative sign, we get

$$x = \frac{2r^3 - a^2r - a^2r}{a^2 - r^2} = \frac{2r^3 - 2a^2r}{a^2 - r^2} = -2r$$

But, the distance x can't be negative i.e. $x > 0$ hence this value is discarded. Thus we get

$$x = \frac{2r^3}{a^2 - r^2}$$

Hence the length of other leg BC of right ΔABC is given as

$$BC = BD + DC = 2r + x$$

$$BC = 2r + \frac{2r^3}{a^2 - r^2}$$

$$BC = \frac{2a^2r - 2r^3 + 2r^3}{a^2 - r^2}$$

$$BC = \frac{2a^2r}{a^2 - r^2} \qquad \textbf{Proved}$$

18. In a square ABCD of each side a, all the vertices are joined to the mid-points of their opposite sides to obtain a small square PQRS (as shown in the fig-33). The length of each side & the area of square PQRS, are given by the following formula

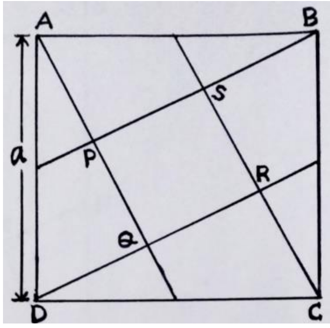

Fig-33: A square PQRS is obtained by joining vertices to mid-points of opposite sides of square ABCD

$$PQ = \frac{a}{\sqrt{5}} \quad \& \quad \textbf{Area of Square PQRS} = \frac{1}{5}(\textbf{Area of Square ABCD})$$

Proof: Consider a square $ABCD$ with each side a. Now join the vertices A, B, C & D to the respective mid-points G, H, E & F of their opposite sides to obain small square PQRS . (As shown in the fig-34). Let $PQ = x$ be each side of square PQRS.

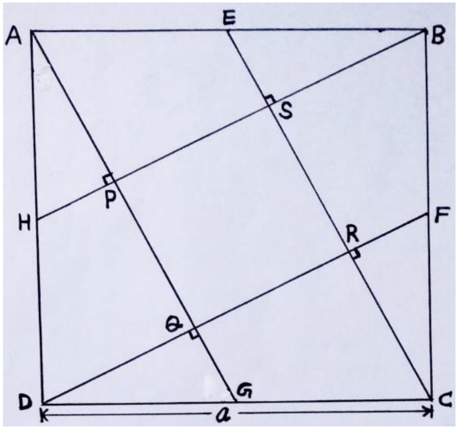

Fig-34: In right ΔAQD, the lines PH & DQ are parallel since the lines BH & DF are parallel to each other. $PQ = QR = RS = PS = x$

Since, in right ΔAQD, the straight lines HP & DQ are parallel hence from **Thales Theorem**, we have

$$\frac{AP}{PQ} = \frac{AH}{HD}$$

$$\frac{AP}{x} = \frac{HD}{HD} \qquad (\text{Since}, \ \ AH = HD)$$

$$AP = x$$

$$\Rightarrow AQ = AP + PQ = x + x = 2x$$

Similarly, in right ΔDRC, the straight lines GQ & CR are parallel hence from **Thales Theorem**, we have

$$\frac{DQ}{QR} = \frac{DG}{GC}$$

$$\frac{DQ}{x} = \frac{GC}{GC} \qquad (\text{Since}, \ \ DG = GC)$$

$$DQ = x$$

Using Pythagoras theorem in right ΔAQD (See above fig-34),

$$AQ^2 + DQ^2 = AD^2$$

$$(2x)^2 + x^2 = a^2$$

$$4x^2 + x^2 = a^2$$

$$x^2 = \frac{a^2}{5}$$

$$x = \frac{a}{\sqrt{5}} \qquad \textbf{Proved}$$

Hence, the **area of square PQRS** of each side $PQ = x$, is

$$= x^2$$

$$= \left(\frac{a}{\sqrt{5}}\right)^2$$

$$= \frac{a^2}{5}$$

$$= \frac{1}{5}(\text{Area of square ABCD}) \qquad \textbf{Proved}$$

19. Four quarter-circles each with a radius a & centre at vertex of a square ABCD of each side a , are drawn which intersect one another at four distinct points P, Q, R & S (as shown in the fig-35). The length of each side of square PQRS, the angle exerted by side PQ at the vertex D & the area A bounded by four quarter-circles are given by the following formula

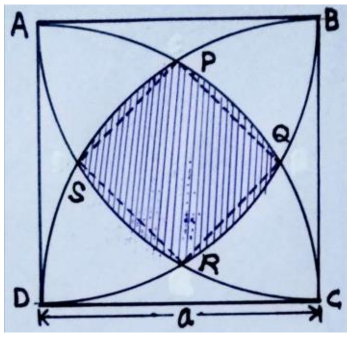

Fig-35: Four quarter-circles each with a radius a & centre at vertex, intersect one another at P, Q, R, S

$$\boldsymbol{PQ = \frac{a}{2}\left(\sqrt{6}-\sqrt{2}\right), \ \angle PDQ = \frac{\pi}{6} \ \ \& \ \ A = \left(\frac{\pi}{3}+1-\sqrt{3}\right)a^2}$$

Proof: Consider a square ABCD of each side a such that its vetrex D is at the origin O in XY-plane & the sides CD & AD coincide with X & Y axes respectively. Now, draw four identical quarter-circles each with a radius a & center at a vertex of square $ABCD$ which intersect one another at four distinct points P, Q, R & S. Now join the vertices P, Q, R & S by dotted lines & drop a perpendicuar OM to the side PQ. (As shown in the fig-36).

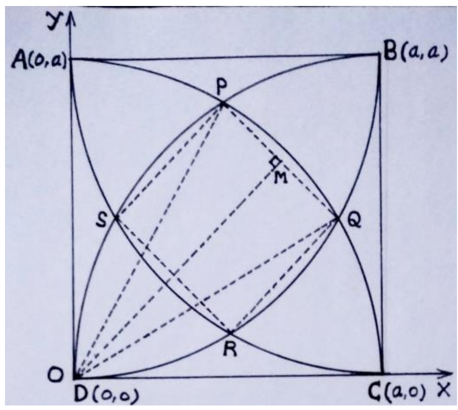

Fig-36: Square ABCD of each side a is drawn in XY-plane such that its vertex D is at origin O and sides CD & AD coincide with x & y axes

Equation of circle with center at the origin O i.e. $D(0,0)$ & radius a is given as

$$x^2 + y^2 = a^2 \qquad \dots\dots\dots\dots\dots (1)$$

Similarly, equation of circle with center at vertex $A(0,a)$ & radius a is given as

$$x^2 + (y-a)^2 = a^2$$

$$x^2 + y^2 + a^2 - 2ay = a^2$$

$$x^2 + y^2 - 2ay = 0 \qquad \dots\dots\dots\dots\dots (2)$$

Similarly, equation of circle with center at vertex $C(a,0)$ & radius a is given as

$$(x-a)^2 + y^2 = a^2$$

$$x^2 + a^2 - 2ax + y^2 = a^2$$

$$x^2 + y^2 - 2ax = 0 \qquad \dots\dots\dots\dots\dots (3)$$

Now, solve (1) & (2) by subtituting value of x^2+y^2 from (1) into (2) as follows

$$a^2-2ay=0 \quad \Rightarrow y=\frac{a}{2}$$

Substituting $y=\frac{a}{2}$ in (1) as follows

$$x^2+\left(\frac{a}{2}\right)^2=a^2 \quad \Rightarrow \quad x=\frac{a\sqrt{3}}{2}$$

Thus, we get coordinates of point of intersection Q as follows

$$Q\equiv\left(\frac{a\sqrt{3}}{2},\frac{a}{2}\right)$$

Similarly, solve (1) & (3) by subtituting value of x^2+y^2 from (1) into (3) as follows

$$a^2-2ax=0 \quad \Rightarrow x=\frac{a}{2}$$

Substituting $x=\frac{a}{2}$ in (1) as follows

$$\left(\frac{a}{2}\right)^2+y^2=a^2 \quad \Rightarrow \quad y=\frac{a\sqrt{3}}{2}$$

Thus, we get coordinates of point of intersection P as follows

$$P\equiv\left(\frac{a}{2},\frac{a\sqrt{3}}{2}\right)$$

Now, using distance formula, the length of line PQ with end points $P\equiv\left(\frac{a}{2},\frac{a\sqrt{3}}{2}\right)$ & $Q\equiv\left(\frac{a\sqrt{3}}{2},\frac{a}{2}\right)$ is given as

$$PQ=\sqrt{\left(\frac{a\sqrt{3}}{2}-\frac{a}{2}\right)^2+\left(\frac{a}{2}-\frac{a\sqrt{3}}{2}\right)^2}$$

$$PQ=\frac{a}{2}\sqrt{\left(\sqrt{3}-1\right)^2+\left(\sqrt{3}-1\right)^2}$$

$$PQ=\frac{a\left(\sqrt{3}-1\right)\sqrt{2}}{2}$$

$$PQ=\frac{a}{2}\left(\sqrt{6}-\sqrt{2}\right) \qquad \textbf{Proved}$$

Now, in right ΔDMP (see above fig-36)

$$\sin\angle PDM=\frac{PM}{DP}=\frac{PQ/2}{a}=\frac{PQ}{2a}=\frac{\frac{a}{2}\left(\sqrt{6}-\sqrt{2}\right)}{2a}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\Rightarrow \cos\angle PDM=\sqrt{1-\sin^2\angle PDM}=\sqrt{1-\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2}=\frac{\sqrt{2+\sqrt{3}}}{2}$$

$$\therefore \sin\angle PDQ=\sin 2\angle PDM \qquad (\text{since, } \angle \text{PDQ}=2\angle \text{PDM})$$

$$\sin\angle PDQ = 2\sin\angle PDM\cos\angle PDM$$

$$\sin\angle PDQ = 2\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\left(\frac{\sqrt{2+\sqrt{3}}}{2}\right)$$

$$\sin\angle PDQ = \frac{\sqrt{\left(\sqrt{6}-\sqrt{2}\right)^2\left(2+\sqrt{3}\right)}}{4}$$

$$\sin\angle PDQ = \frac{\sqrt{\left(6+2-4\sqrt{3}\right)\left(2+\sqrt{3}\right)}}{4}$$

$$\sin\angle PDQ = \frac{\sqrt{4\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)}}{4}$$

$$\sin\angle PDQ = \frac{2\sqrt{4-3}}{4}$$

$$\sin\angle PDQ = \frac{1}{2}$$

$$\angle PDQ = \frac{\pi}{6} \qquad \textbf{proved}$$

Now, the area of segment of circle bounded by the arc PQ & chord PQ (See above fig-36) is

$$= \text{Area of sector DPQD} - \text{Area of isosceles } \Delta\text{PDQ}$$

$$= \frac{1}{2}(\angle PDQ)a^2 - \frac{1}{2}a^2\sin\angle PDQ$$

$$= \frac{1}{2}\left(\frac{\pi}{6}\right)a^2 - \frac{1}{2}a^2\sin\frac{\pi}{6}$$

$$= \frac{1}{2}\left(\frac{\pi}{6}\right)a^2 - \frac{1}{2}a^2\left(\frac{1}{2}\right)$$

$$= \frac{1}{4}\left(\frac{\pi}{3}-1\right)a^2$$

Hence, the area of region bounded by four intersecting quarter-circles (see above fig-36) is

$$= 4(\text{Area of segment bounded by arc PQ \& } chord\ PQ) + \text{Area of square PQRS}$$

$$= 4\left(\frac{1}{4}\left(\frac{\pi}{3}-1\right)a^2\right) + \left(\frac{a}{2}\left(\sqrt{6}-\sqrt{2}\right)\right)^2$$

$$= \left(\frac{\pi}{3}-1\right)a^2 + \frac{1}{4}\left(6+2-4\sqrt{3}\right)a^2$$

$$= \left(\frac{\pi}{3}-1\right)a^2 + \left(2-\sqrt{3}\right)a^2$$

$$= \left(\frac{\pi}{3}-1+2-\sqrt{3}\right)a^2$$

$$= \left(\frac{\pi}{3} + 1 - \sqrt{3}\right)a^2 \qquad \textbf{proved}$$

20. A quarter-circle with a radius a & centre at vertex A & a semi-circle with a radius $a/2$ & centre at mid-point of side CD of square ABCD of each side a, are drawn which intersect each other at two distinct points (as shown in the fig-37). The area A (as shaded in fig-37) bounded by quarter-circle, semicircle & diagonal BD is given by the following formula

$$\mathbf{A = \frac{a^2}{8}\left(2\pi - 3 - 3\sin^{-1}\left(\frac{3}{5}\right)\right)}$$

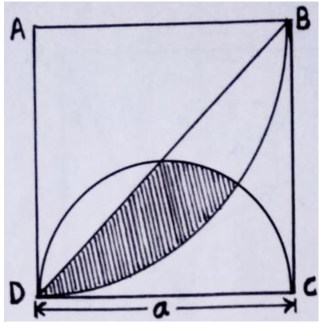

Fig-37: The shaded region is bounded by a quarter-circle, a semi-circle & diagonal BD

Proof: Consider a square ABCD of each side a such that its vetrex D is at the origin O in XY-plane & the sides CD & AD coincide with X & Y axes respectively. Now, draw a quarter-circle with a radius a & center at a vertex A & a semi-circle with a radius $a/2$ & centre at mid-point M of side CD of square $ABCD$ which intersect each other at two distinct points O & Q. Now drop a perpendicuar PM from P to the side CD which divides the bounded area into two parts. (As shown in the fig-38).

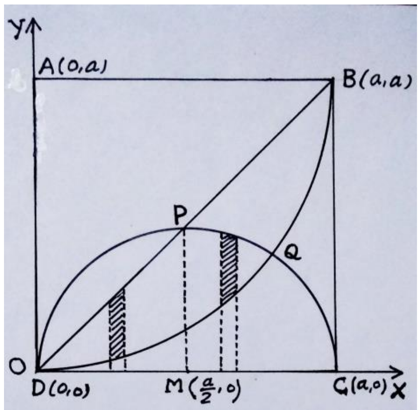

Fig-38: Two elementary rectangular slabs sweep the area bounded by a semi-circle, a quarter circle & diagonal BD in square ABCD of each side a

Equation of circle with center at the mid-point $M\left(\frac{a}{2}, 0\right)$ & radius $\frac{a}{2}$ is given as

$$\left(x - \frac{a}{2}\right)^2 + (y - 0)^2 = \left(\frac{a}{2}\right)^2$$

$$x^2 + y^2 - ax = 0 \qquad \dots\dots\dots\dots\dots (1)$$

Similarly, equation of circle with center at vertex $A(0, a)$ & radius a is given as

$$x^2 + (y - a)^2 = a^2$$

$$x^2 + y^2 + a^2 - 2ay = a^2$$

$$x^2 + y^2 - 2ay = 0 \qquad \dots\dots\dots\dots\dots (2)$$

Substituting $x^2 + y^2 = ax$ from (1) into (2) as follows

$$ax - 2ay = 0 \quad \Rightarrow y = \frac{x}{2}$$

Now, substituting $y = x/2$ into (1) as follows

$$x^2 + \left(\frac{x}{2}\right)^2 - ax = 0$$

$$5x^2 - 4ax = 0 \;\Rightarrow\; x = 0, \frac{4a}{5} \;\Rightarrow\; y = 0, \frac{2a}{5}$$

$$\therefore P \equiv \left(\frac{a}{2}, \frac{a}{2}\right) \;\;\&\;\; Q \equiv \left(\frac{4a}{5}, \frac{2a}{5}\right)$$

Now, divide the bounded region into two parts out of which (See above fig-38 above)

1. One part is bounded by the diagonal BD: $y = x$ $(\forall\, 0 \le x, y \le a)$ & the quarter circle: $y = a - \sqrt{a^2 - x^2}$ $(\forall\, 0 \le x, y \le a)$ from eq(2) &

2. Other part is bounded by semi-circle: $y = \sqrt{ax - x^2}$ $(\forall\, 0 \le x \le a,\ 0 \le y \le a/2)$ from eq(1) & the quarter circle: $y = a - \sqrt{a^2 - x^2}$ $(\forall\, 0 \le x, y \le a)$ from eq(2)

Consider two vertical elementary rectangular slabs to compute the area bounded by a semi-circle, a quarter-circle & diagonal BD (as shown in the fig-38 above). Using integration with proper limits, the bounded area A is given as follows

$$A = \int_0^{a/2} \left(x - \left(a - \sqrt{a^2 - x^2}\right)\right) dx + \int_{a/2}^{4a/5} \left(\sqrt{ax - x^2} - \left(a - \sqrt{a^2 - x^2}\right)\right) dx$$

$$= \int_0^{a/2} (x - a) dx + \int_0^{a/2} \sqrt{a^2 - x^2}\, dx + \int_{a/2}^{4a/5} \left(\sqrt{ax - x^2} - a\right) dx + \int_{a/2}^{4a/5} \sqrt{a^2 - x^2}\, dx$$

$$= \int_0^{a/2} (x - a) dx + \int_{a/2}^{4a/5} \left(\sqrt{ax - x^2} - a\right) dx + \int_0^{4a/5} \sqrt{a^2 - x^2}\, dx$$

$$= \left[\frac{x^2}{2} - ax\right]_0^{a/2} + \int_{a/2}^{4a/5} \left(\sqrt{\left(\frac{a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2} - a\right) dx + \int_0^{4a/5} \sqrt{a^2 - x^2} dx$$

$$= \left[\frac{a^2}{8} - \frac{a^2}{2}\right] + \left[\frac{1}{2}\left(x - \frac{a}{2}\right)\sqrt{\left(\frac{a}{2}\right)^2 - \left(x - \frac{a}{2}\right)^2} + \frac{1}{2}\left(\frac{a}{2}\right)^2 \sin^{-1}\left(\frac{x - \frac{a}{2}}{\frac{a}{2}}\right) - ax\right]_{a/2}^{4a/5}$$
$$+ \left[\frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]_0^{4a/5}$$

$$= \frac{-3a^2}{8} + \left[\frac{1}{2}\left(\frac{4a}{5} - \frac{a}{2}\right)\sqrt{\left(\frac{a}{2}\right)^2 - \left(\frac{4a}{5} - \frac{a}{2}\right)^2} + \frac{1}{2}\left(\frac{a}{2}\right)^2 \sin^{-1}\left(\frac{\frac{4a}{5} - \frac{a}{2}}{\frac{a}{2}}\right) - a\left(\frac{4a}{5}\right) + a\left(\frac{a}{2}\right)\right]$$
$$+ \left[\frac{1}{2}\left(\frac{4a}{5}\right)\sqrt{a^2 - \left(\frac{4a}{5}\right)^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{\frac{4a}{5}}{a}\right)\right]$$

$$= \frac{-3a^2}{8} + \left[\frac{1}{2}\left(\frac{3a}{10}\right)\left(\frac{2a}{5}\right) + \frac{a^2}{8}\sin^{-1}\left(\frac{3}{5}\right) - \frac{3a^2}{10}\right] + \left[\frac{1}{2}\left(\frac{4a}{5}\right)\left(\frac{3a}{5}\right) + \frac{a^2}{2}\sin^{-1}\left(\frac{4}{5}\right)\right]$$

$$= \frac{-3a^2}{8} - \frac{6a^2}{25} + \frac{a^2}{8}\sin^{-1}\left(\frac{3}{5}\right) + \frac{6a^2}{25} + \frac{a^2}{2}\sin^{-1}\left(\frac{4}{5}\right)$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{4}{5}\right) + \frac{a^2}{8}\sin^{-1}\left(\frac{3}{5}\right) - \frac{3a^2}{8}$$

$$= \frac{a^2}{8}\left(4\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{3}{5}\right) - 3\right)$$

$$= \frac{a^2}{8}\left(4\sin^{-1}\left(\frac{4}{5}\right) + 4\sin^{-1}\left(\frac{3}{5}\right) - 3\sin^{-1}\left(\frac{3}{5}\right) - 3\right)$$

$$= \frac{a^2}{8}\left(4\left(\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{3}{5}\right)\right) - 3 - 3\sin^{-1}\left(\frac{3}{5}\right)\right)$$

$$=\frac{a^2}{8}\left(4\left(\sin^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{4}{5}\right)\right)-3-3\sin^{-1}\left(\frac{3}{5}\right)\right) \qquad \left(\text{since,}\quad \sin^{-1}\left(\frac{3}{5}\right)=\cos^{-1}\left(\frac{4}{5}\right)\right)$$

$$=\frac{a^2}{8}\left(4\left(\frac{\pi}{2}\right)-3-3\sin^{-1}\left(\frac{3}{5}\right)\right) \qquad \left(\text{since,}\quad \sin^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{4}{5}\right)=\frac{\pi}{2}\right)$$

$$=\frac{a^2}{8}\left(2\pi-3-3\sin^{-1}\left(\frac{3}{5}\right)\right) \qquad \textbf{proved}$$

21. If d is the distance between the centres A & B of two circles having radii r_1 & r_2 respectively (as shown in the fig-39) then the lengths of open common tangent PQ & cross common tangent RS are given by the following formula

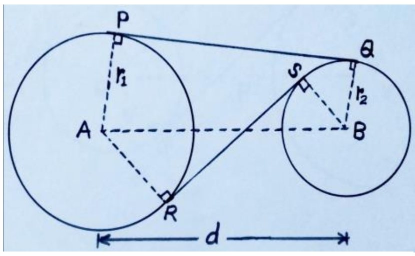

Fig-39: Open tangent PQ & cross tangent RS are drawn on two circles with radii r_1 & r_2 at a distance d b/w their centres

$$\mathbf{PQ}=\sqrt{\boldsymbol{d}^2-(\boldsymbol{r}_1-\boldsymbol{r}_2)^2}\quad \forall\ \boldsymbol{d}>|\boldsymbol{r}_1-\boldsymbol{r}_2|$$ **&**

$$\mathbf{RS}=\sqrt{\boldsymbol{d}^2-(\boldsymbol{r}_1+\boldsymbol{r}_2)^2}\quad \forall\ \boldsymbol{d}>(\boldsymbol{r}_1+\boldsymbol{r}_2)$$

Both open & cross common tangents can be simultaneously drawn only if $d>(r_1+r_2)$

Proof: Consider two circles with centres A & B and radii r_1& r_2 respectively at a distance d between their centres A & B. Now, draw the open common tangent PQ to the circles & join the points P and Q to the centres A & B respectively by dotted straight lines. Join the centres A & B by a dotted straight line & drop the perpendicular BM from centre B to the radial line PA (As shown in the fig-40). Thus we get a rectangle PQBM in which

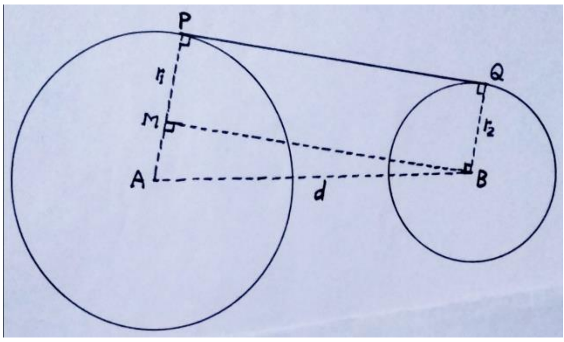

Fig-40: A Perpendicular BM is drawn from B to radial line AP such that $PM=QB=r_2$ & $AM=r_1-r_2$

$$BM=PQ,\quad PM=QB=r_2$$

$$\therefore AM=PA-PM=r_1-r_2$$

Now, using Pythagoras Theorem in right ΔAMB as follows

$$AM^2+BM^2=AB^2$$

$$(r_1-r_2)^2+PQ^2=d^2 \qquad (\text{since,}\quad \text{BM}=\text{PQ})$$

$$PQ^2=d^2-(r_1-r_2)^2$$

$$PQ=\sqrt{d^2-(r_1-r_2)^2} \qquad \textbf{proved}$$

Similarly, consider two circles with centres A & B and radii r_1& r_2 respectively at a distance d between their centres A & B. Now, draw the cross common tangent RS to the circles & join the points R and S to the centres A & B respectively by dotted straight lines. Join the centres A & B by a dotted straight line & drop the perpendicular BM from centre B to the extended radial line AR (As shown in the fig-41). Thus we get a rectangle RSBM in which

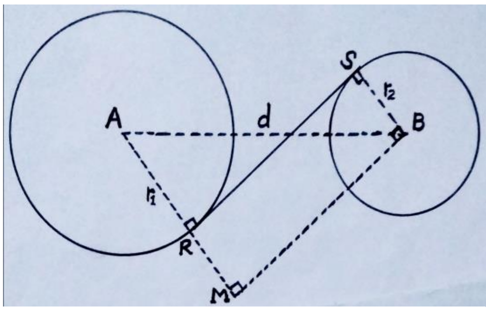

Fig-41: A Perpendicular BM is drawn from centre B to the extended radial line AR such that $RM=SB=r_2$ & $AM=r_1+r_2$

$$BM=RS,\quad RM=SB=r_2$$

$$\therefore AM=AR+RM=r_1+r_2$$

Now, using Pythagoras Theorem in right ΔAMB (see fig-41 above) as follows

$$AM^2 + BM^2 = AB^2$$

$$(r_1 + r_2)^2 + RS^2 = d^2 \qquad (\text{since, } \ \mathrm{BM} = \mathrm{RS})$$

$$RS^2 = d^2 - (r_1 + r_2)^2$$

$$RS = \sqrt{d^2 - (r_1 + r_2)^2} \qquad \textbf{proved}$$

Note: Above articles had been derived *& illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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