$Computation\ of\ Area\ of\ Spherical\ Triangle$

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Introduction: Here, we are to compute the area of the spherical triangle having each side as a great circle arc on the spherical surface when 1.) aperture angle subtended by each of three sides at the centre of sphere are known 2.) arc length of each of three sides is known. We will derive the formula to compute area of spherical triangle in both the above cases.

Case-1. Area of spherical triangle when aperture angles subtended by the sides at the centre of sphere are known: Consider a spherical triangle **ABC** having each side as a great circle arc on a spherical surface of radius $\boldsymbol{R}$ such that $\boldsymbol{\alpha}, \boldsymbol{\beta}\ \&\ \boldsymbol{\gamma}$ are aperture angles subtended at the centre O by the sides (each as a great circle-arc) BC, AC & AB respectively. Join the vertices A, B & C with each other & to the centre O of sphere by the dotted straight lines to get a tetrahedron OABC (As shown by plane-projection in the figure-1).

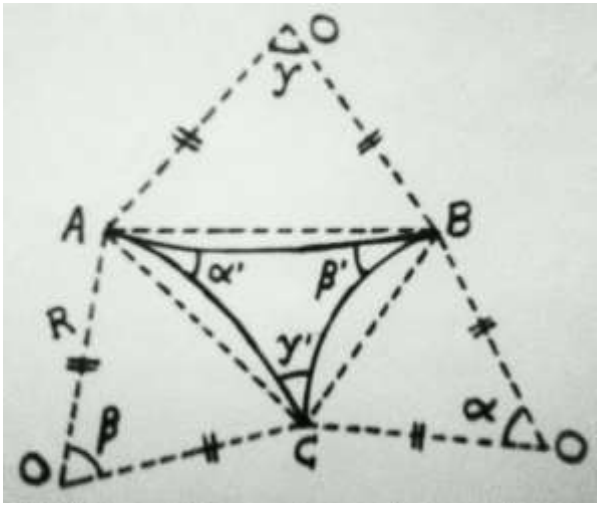

Figure 1: Plane projection of vertices A, B, C & centre O. $\boldsymbol{\alpha}, \boldsymbol{\beta}\ \&\ \boldsymbol{\gamma}$ are aperture angles subtended by the sides (each as a great circle arc) BC, AC & AB respectively of spherical $\Delta \boldsymbol{ABC}$ at the centre O of sphere of radius R. OABC is a tetrahedron obtained by joining vertices A, B & C & centre O by dotted straight lines.

Now, if $\boldsymbol{\alpha}', \boldsymbol{\beta}'\ \&\ \boldsymbol{\gamma}'$ are the interior angles of spherical ΔABC opposite to the aperture angles $\boldsymbol{\alpha}, \boldsymbol{\beta}\ \&\ \boldsymbol{\gamma}$ respectively then by using **Cosine Formula for tetrahedron OABC**, interior angle $\boldsymbol{\alpha}'$ of spherical triangle opposite to aperture angle $\boldsymbol{\alpha}$ is given as

$$\cos\alpha' = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}$$

Or

$$\boldsymbol{\alpha}' = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right)$$

Similarly, angle $\boldsymbol{\beta}'$ opposite to the angle $\boldsymbol{\beta}$ is given as

$$\boldsymbol{\beta}' = \cos^{-1}\left(\frac{cos\,\beta - cos\,\alpha\,cos\,\gamma}{sin\,\alpha\,sin\,\gamma}\right)$$

Similarly, angle $\boldsymbol{\gamma}'$ opposite to the angle $\boldsymbol{\gamma}$ is given as

$$\boldsymbol{\gamma}' = \cos^{-1}\left(\frac{cos\,\gamma - cos\,\alpha\,cos\,\beta}{sin\,\alpha\,sin\,\beta}\right)$$

Now, the area of spherical triangle ABC having interior angles $\alpha', \beta'\ \&\ \gamma'$ on the spherical surface of radius R is given as

$$\boldsymbol{Area\ of\ spherical}\ \Delta\boldsymbol{ABC} = (\boldsymbol{\alpha}' + \boldsymbol{\beta}' + \boldsymbol{\gamma}' - \boldsymbol{\pi})\boldsymbol{R}^2$$

Case-2. Area of spherical triangle when arc lengths of all three sides are known: Consider a spherical triangle **ABC** having each of sides BC, AC & AB as a great circle arc of lengths $\boldsymbol{a}, \boldsymbol{b}\ \&\ \boldsymbol{c}$ respectively on a spherical surface of radius $\boldsymbol{R}$. Now, aperture angles $\boldsymbol{\alpha}, \boldsymbol{\beta}\ \&\ \boldsymbol{\gamma}$ subtended at the centre O by the sides (each as a great circle-arc) $BC = a$, $AC = b$ & $AB = c$ respectively are given as (See the above figure-1 for clarification)

$$\alpha = \frac{\text{great arc length BC}}{\text{radius of sphere}} = \frac{a}{R}$$

Similarly,

$$\beta = \frac{\text{great arc AC}}{R} = \frac{b}{R}, \quad \gamma = \frac{\text{great arc AB}}{R} = \frac{c}{R}$$

Now, setting the values of α, β & γ in the formula of case-1, the interior angles α', β' & γ' opposite to the sides $BC = a, AC = b$ & $AB = c$ respectively of spherical ΔABC on a sphere of radius R are given as (see above fig1)

$$\boldsymbol{\alpha'} = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) = \mathbf{cos^{-1}}\left(\frac{\mathbf{cos}\frac{\boldsymbol{a}}{\boldsymbol{R}} - \mathbf{cos}\frac{\boldsymbol{b}}{\boldsymbol{R}}\mathbf{cos}\frac{\boldsymbol{c}}{\boldsymbol{R}}}{\mathbf{sin}\frac{\boldsymbol{b}}{\boldsymbol{R}}\mathbf{sin}\frac{\boldsymbol{c}}{\boldsymbol{R}}}\right)$$

$$\boldsymbol{\beta'} = \cos^{-1}\left(\frac{cos\,\beta - cos\,\alpha\,cos\,\gamma}{sin\,\alpha\,sin\,\gamma}\right) = \mathbf{cos^{-1}}\left(\frac{\mathbf{cos}\frac{\boldsymbol{b}}{\boldsymbol{R}} - \mathbf{cos}\frac{\boldsymbol{a}}{\boldsymbol{R}}\mathbf{cos}\frac{\boldsymbol{c}}{\boldsymbol{R}}}{\mathbf{sin}\frac{\boldsymbol{a}}{\boldsymbol{R}}\mathbf{sin}\frac{\boldsymbol{c}}{\boldsymbol{R}}}\right)$$

$$\boldsymbol{\gamma'} = \cos^{-1}\left(\frac{cos\,\gamma - cos\,\alpha\,cos\,\beta}{sin\,\alpha\,sin\,\beta}\right) = \mathbf{cos^{-1}}\left(\frac{\mathbf{cos}\frac{\boldsymbol{c}}{\boldsymbol{R}} - \mathbf{cos}\frac{\boldsymbol{a}}{\boldsymbol{R}}\mathbf{cos}\frac{\boldsymbol{b}}{\boldsymbol{R}}}{\mathbf{sin}\frac{\boldsymbol{a}}{\boldsymbol{R}}\mathbf{sin}\frac{\boldsymbol{b}}{\boldsymbol{R}}}\right)$$

Now, the area of spherical triangle ABC having interior angles α', β' & γ' on the spherical surface of radius R is given as

$$\boldsymbol{Area\ of\ spherical\ \Delta ABC = (\alpha' + \beta' + \gamma' - \pi)R^2}$$

Illustrative Examples

Q1. Compute the area of spherical triangle on a sphere of radius 15 cm such that $\mathbf{29^o, 50^o}$ & $\mathbf{65^o}$ are the aperture angles subtended by the sides (each as a great circle arc) at the centre of sphere.

Ans. The interior angles α', β' & γ' of a spherical triangle opposite to the aperture angles $\alpha = 29^o, \beta = 50^o$ & $\gamma = 65^o$ respectively subtended by the sides at the centre of a sphere are computed by using general formula (above case-1) as follows

$$\alpha' = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) = \cos^{-1}\left(\frac{\cos 29^o - \cos 50^o \cos 65^o}{\sin 50^o \sin 65^o}\right) = 29^o 43' 0.34'' = 0.51865532 \text{ rad.}$$

$$\beta' = \cos^{-1}\left(\frac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}\right) = \cos^{-1}\left(\frac{\cos 50^o - \cos 29^o \cos 65^o}{\sin 29^o \sin 65^o}\right) = 51^o 33' 40.24'' = 0.89991232 \text{ rad.}$$

$$\gamma' = \cos^{-1}\left(\frac{cos\,\gamma - cos\,\alpha\,cos\,\beta}{sin\,\alpha\,sin\,\beta}\right) = \cos^{-1}\left(\frac{\cos 65^o - \cos 29^o \cos 50^o}{\sin 29^o \sin 50^o}\right) = 112^o 4.52' 31.37''$$
$$= 1.956084404 \text{ rad.}$$

Hence, the area of spherical triangle ABC having interior angles α', β' & γ' on the spherical surface of radius $R = 15\ cm$ is

$$= (\alpha' + \beta' + \gamma' - \pi)R^2 = (0.51865532 + 0.89991232 + 1.956084404 - \pi)15^2 = \mathbf{52.43836295\ cm^2}$$

Q2. Compute the area of spherical triangle having each side as a great circle arc of lengths $6\ cm, 8\ cm, 10\ cm$ on a sphere of radius 24 cm.

Ans. The interior angles $\alpha', \beta'\ \&\ \gamma'$ of a spherical triangle opposite to the sides (each as a great circle arc) $a = 6\ cm, b = 8\ cm\ \&\ c = 10\ cm$ respectively on a sphere of radius $R = 24\ cm$ are computed by using general formula (above case-2) as follows

$$\alpha' = \cos^{-1}\left(\frac{\cos\frac{a}{R} - \cos\frac{b}{R}\cos\frac{c}{R}}{\sin\frac{b}{R}\sin\frac{c}{R}}\right) = \cos^{-1}\left(\frac{\cos\frac{6}{24} - \cos\frac{8}{24}\cos\frac{10}{24}}{\sin\frac{8}{24}\sin\frac{10}{24}}\right) = 0.65763203 \text{ rad.}$$

$$\beta' = \cos^{-1}\left(\frac{\cos\frac{b}{R} - \cos\frac{a}{R}\cos\frac{c}{R}}{\sin\frac{a}{R}\sin\frac{c}{R}}\right) = \cos^{-1}\left(\frac{\cos\frac{8}{24} - \cos\frac{6}{24}\cos\frac{10}{24}}{\sin\frac{6}{24}\sin\frac{10}{24}}\right) = 0.941391672 \text{ rad.}$$

$$\gamma' = \cos^{-1}\left(\frac{\cos\frac{c}{R} - \cos\frac{a}{R}\cos\frac{b}{R}}{\sin\frac{a}{R}\sin\frac{b}{R}}\right) = \cos^{-1}\left(\frac{\cos\frac{10}{24} - \cos\frac{6}{24}\cos\frac{8}{24}}{\sin\frac{6}{24}\sin\frac{8}{24}}\right) = 1.584848266 \text{ rad.}$$

Hence, the area of spherical triangle ABC having interior angles $\alpha', \beta'\ \&\ \gamma'$ on the spherical surface of radius $R = 24\ cm$ is

$$= (\alpha' + \beta' + \gamma' - \pi)R^2 = (0.65763203 + 0.941391672 + 1.584848266 - \pi)24^2 = \mathbf{24.35288494\ cm^2}$$

Important note: In any spherical ΔABC drawn on a sphere of radius R, the interior angles $A, B\ \&\ C$ opposite to the sides $BC = a, AC = b, AB = c$ (each as a great circle arc) respectively are given by the general formula (as derived in above case-2)

$$A = \cos^{-1}\left(\frac{\cos\frac{a}{R} - \cos\frac{b}{R}\cos\frac{c}{R}}{\sin\frac{b}{R}\sin\frac{c}{R}}\right),\quad B = \cos^{-1}\left(\frac{\cos\frac{b}{R} - \cos\frac{a}{R}\cos\frac{c}{R}}{\sin\frac{a}{R}\sin\frac{c}{R}}\right)\ \&\ C = \cos^{-1}\left(\frac{\cos\frac{c}{R} - \cos\frac{a}{R}\cos\frac{b}{R}}{\sin\frac{a}{R}\sin\frac{b}{R}}\right)$$

Note: Above articles had been developed & *illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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