$Proofs\ of\ Apollonius\ Theorem$

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Here, we are interested to prove Apollonius Theorem by three different methods 1. using Trigonometry and 2. using Pythagoras Theorem. Apollonius theorem basically correlates lengths of all three sides & one median in a triangle.

Apollonius Theorem: In any triangle, the sum of squares of any two sides is equal to the sum of half the square of third side and twice the square of corresponding median i.e. if a, b, c are three sides BC, AC and AB respectively & m is the length of median in a triangle ΔABC (As shown in the fig-1) then

$$b^2 + c^2 = \frac{a^2}{2} + 2m^2$$

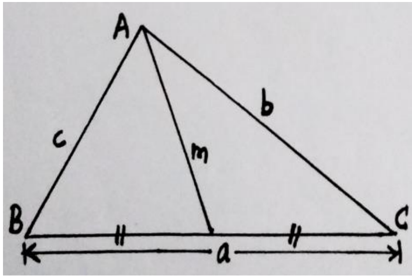

Fig-1: ΔABC having sides a, b, c & a median of length m

Proof-1 (Using Trigonometry): Consider a triangle ΔABC having sides BC, AC and AB of lengths a, b & c respectively and the median AD of length m. (as shown in the fig-2)

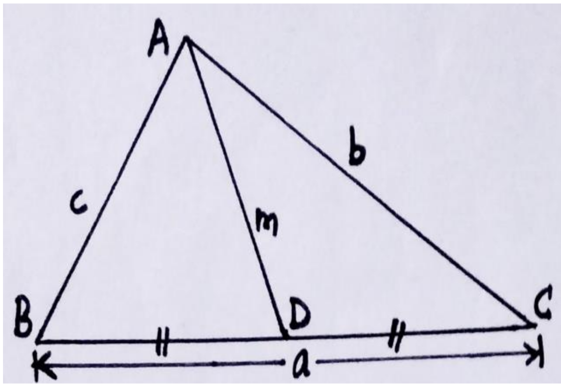

Fig-2: In ΔABC, $BD = CD = a/2$ & $AD = m$

Now, applying Cosine rule in ΔABD (See fig-2)

$$\cos B = \frac{AB^2 + BD^2 - AD^2}{2(AB)(BD)}$$

$$\cos B = \frac{c^2 + \left(\frac{a}{2}\right)^2 - m^2}{2(c)\left(\frac{a}{2}\right)}$$

$$ac \cos B = c^2 + \frac{a^2}{4} - m^2 \quad \dots\dots\dots\dots\dots(1)$$

Again, applying Cosine rule in ΔACD

$$\cos C = \frac{AC^2 + CD^2 - AD^2}{2(AC)(CD)}$$

$$\cos C = \frac{b^2 + \left(\frac{a}{2}\right)^2 - m^2}{2(b)\left(\frac{a}{2}\right)}$$

$$ab \cos C = b^2 + \frac{a^2}{4} - m^2 \quad \dots\dots\dots\dots\dots(2)$$

Adding (1) & (2), we get

$$ac \cos B + ab \cos C = c^2 + \frac{a^2}{4} - m^2 + b^2 + \frac{a^2}{4} - m^2$$

$$a(b\cos C + c\cos B) = \frac{a^2}{2} + b^2 + c^2 - 2m^2$$

Setting $b = K\sin B$ & $c = K\sin C$ from Sine rule in ΔABC,

$$a(K\sin B\ \cos C + K\sin C\cos B) = \frac{a^2}{2} + b^2 + c^2 - 2m^2$$

$$aK(\sin B\ \cos C + \sin C\cos B) = \frac{a^2}{2} + b^2 + c^2 - 2m^2$$

$$aK\sin(B + C) = \frac{a^2}{2} + b^2 + c^2 - 2m^2$$

$$aK\sin(\pi - A) = \frac{a^2}{2} + b^2 + c^2 - 2m^2 \qquad (since,\ \ A + B + C = \pi)$$

$$aK\sin A = \frac{a^2}{2} + b^2 + c^2 - 2m^2$$

$$a \cdot a = \frac{a^2}{2} + b^2 + c^2 - 2m^2 \qquad (from\ Sine\ rule,\ \ a = K\sin A)$$

$$a^2 = \frac{a^2}{2} + b^2 + c^2 - 2m^2$$

$$b^2 + c^2 = \frac{a^2}{2} + 2m^2$$

Proved

Proof-2 (Using Trigonometry): Consider a triangle ΔABC having sides BC, AC and AB of lengths a, b & c respectively and the median AD of length m . Let $\angle BAD = \alpha, \angle CAD = \beta$ & $\angle ADB = \theta$ (as shown in the fig-3)

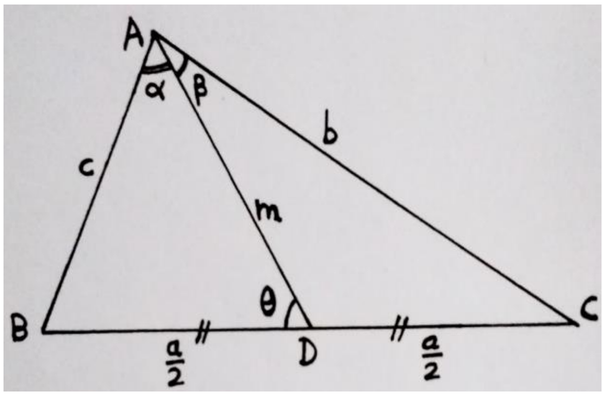

Fig-3: In $\Delta ABC, \angle BAD = \alpha, \angle CAD = \beta$ & $\angle ADB = \theta$

Applying sine rule in ΔABD (See fig-3)

$$\frac{\sin\alpha}{a/2} = \frac{\sin\theta}{c}$$

$$\sin\theta = \frac{2c}{a}\sin\alpha \qquad \dots\dots\dots\dots(1)$$

Applying sine rule in ΔACD

$$\frac{\sin\beta}{a/2} = \frac{\sin(\pi - \theta)}{b}$$

$$\sin\theta = \frac{2b}{a}\sin\beta \qquad \dots\dots\dots\dots(2)$$

Equating values of $\sin\theta$ from (1) & (2), we get

$$\frac{2c}{a}\sin\alpha = \frac{2b}{a}\sin\beta$$

$$\sin\beta = \frac{c}{b}\sin\alpha \qquad \dots\dots\dots\dots\dots(3)$$

Applying cosine rule in ΔABD (See above fig-3)

$$\cos\alpha = \frac{m^2 + c^2 - \left(\frac{a}{2}\right)^2}{2mc} = \frac{4m^2 + 4c^2 - a^2}{8mc} \qquad \dots\dots\dots\dots\dots(4)$$

Applying cosine rule in ΔACD

$$\cos\beta = \frac{m^2 + b^2 - \left(\frac{a}{2}\right)^2}{2mb} = \frac{4m^2 + 4b^2 - a^2}{8mb} \qquad \dots\dots\dots\dots\dots(5)$$

Applying cosine rule in ΔABC

$$\cos(\alpha + \beta) = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{b^2 + c^2 - a^2}{2bc}$$

Setting the values of $\sin\beta$, $\cos\alpha$ & $\cos\beta$ from (3), (4) & (5) respectively as follows

$$\left(\frac{4m^2 + 4c^2 - a^2}{8mc}\right)\left(\frac{4m^2 + 4b^2 - a^2}{8mb}\right) - \sin\alpha\left(\frac{c}{b}\sin\alpha\right) = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\frac{(4m^2 + 4c^2 - a^2)(4m^2 + 4b^2 - a^2)}{64m^2bc} - \frac{c}{b}sin^2\,\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\frac{(4m^2 - a^2 + 4c^2)(4m^2 - a^2 + 4b^2)}{64m^2bc} - \frac{c}{b}(1 - cos^2\,\alpha) - \frac{b^2 + c^2 - a^2}{2bc} = 0$$

$$\frac{(4m^2 - a^2)^2 + 4(b^2 + c^2)(4m^2 - a^2) + 16b^2c^2}{64m^2bc} - \frac{c}{b}\left(1 - \left(\frac{4m^2 + 4c^2 - a^2}{8mc}\right)^2\right) - \frac{b^2 + c^2 - a^2}{2bc} = 0$$

$$\frac{(4m^2 - a^2)^2 + 4(b^2 + c^2)(4m^2 - a^2) + 16b^2c^2}{64m^2bc} - \frac{c}{b}\left(\frac{64m^2c^2 - (4m^2 - a^2)^2 - 16c^4 - 8c^2(4m^2 - a^2)}{64m^2c^2}\right) - \frac{b^2 + c^2 - a^2}{2bc} = 0$$

$$\frac{(4m^2 - a^2)^2 + 4(b^2 + c^2)(4m^2 - a^2) + 16b^2c^2 - 64m^2c^2 + (4m^2 - a^2)^2 + 16c^4 + 8c^2(4m^2 - a^2)}{64m^2bc} - \frac{b^2 + c^2 - a^2}{2bc} = 0$$

$$\frac{2(4m^2 - a^2)^2 + 4(b^2 + 3c^2)(4m^2 - a^2) + 16b^2c^2 - 64m^2c^2 + 16c^4 - 32m^2(b^2 + c^2 - a^2)}{64m^2bc} = 0$$

$$16m^4 + a^4 - 8m^2a^2 + 8m^2b^2 + 24m^2c^2 - 2a^2b^2 - 6a^2c^2 + 8b^2c^2 - 32m^2c^2 + 8c^4 - 16m^2b^2 - 16m^2c^2 + 16m^2a^2 = 0$$

$$(16m^4 + a^4 + 8m^2a^2) - 8m^2b^2 - 24m^2c^2 - 2a^2b^2 - 6a^2c^2 + 8b^2c^2 + 8c^4 = 0$$

$$(4m^2 + a^2)^2 - 8m^2(b^2 + 3c^2) - 2a^2(b^2 + 3c^2) + 8c^2(b^2 + c^2) = 0$$

$$(4m^2 + a^2)^2 - 2(b^2 + 3c^2)(4m^2 + a^2) + 8c^2(b^2 + c^2) = 0$$

Let $4m^2 + a^2 = x$ then we get a quadratic equation as follows

$$x^2-2(b^2+3c^2)x+8c^2(b^2+c^2)=0$$

$$x=\frac{-\left(-2(b^2+3c^2)\right)\pm\sqrt{\left(-2(b^2+3c^2)\right)^2-4\times1\times8c^2(b^2+c^2)}}{2\times1}$$

$$x=b^2+3c^2\pm\sqrt{b^4+9c^4+6b^2c^2-86b^2c^2-8c^4}$$

$$x=b^2+3c^2\pm\sqrt{b^4+c^4-2b^2c^2}$$

$$x=b^2+3c^2\pm(b^2-c^2)$$

Case-1: Taking positive sign, we get

$$x=b^2+3c^2+b^2-c^2$$

$$\Rightarrow 4m^2+a^2=2(b^2+c^2)$$

Case-2: Taking negative sign, we get

$$x=b^2+3c^2-(b^2-c^2)$$

$$4m^2+a^2=4c^2$$

This case holds only if ΔABC is an isosceles triangle i.e. for $b=c$ but it is not the case i.e. this case does not meet the requirements of a scalene triangle

We accept only case-1 which gives us

$$4m^2+a^2=2(b^2+c^2)$$

$$b^2+c^2=\frac{1}{2}(4m^2+a^2)$$

$$b^2+c^2=\frac{a^2}{2}+2m^2$$

Proved

Proof-3 (Using Pythagoras Theorem): Consider a triangle ΔABC having sides BC, AC and AB of lengths a, b & c respectively and the median AD of length m. Drop a perpendicular AN from vertex A to the side BC (as shown by dotted line AN in the fig-4)

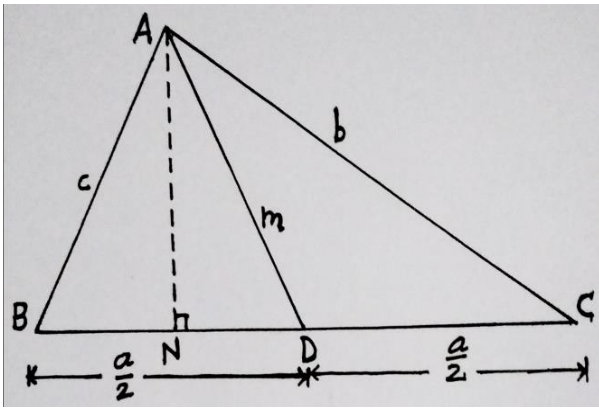

Fig-4: In ΔABC, AD is median of length m & $BD=CD={}^{a}/_{2}$

Applying Pythagoras Theorem in right ΔAND (See fig-4)

$$AN^2+ND^2=AD^2$$

$$AN^2+ND^2=m^2 \qquad \ldots\ldots\ldots.(1)$$

Applying Pythagoras Theorem in right ΔANB

$$AN^2+BN^2=AB^2$$

$$AN^2+\left(\frac{a}{2}-ND\right)^2=c^2$$

$$AN^2 + ND^2 + \frac{a^2}{4} - a(ND) = c^2$$

$$m^2 + \frac{a^2}{4} - a(ND) = c^2 \qquad \text{(from (1), } AN^2 + ND^2 = m^2\text{)}$$

$$a(ND) = m^2 + \frac{a^2}{4} - c^2 \qquad \ldots\ldots\ldots\ldots(2)$$

Applying Pythagoras Theorem in right ΔANC

$$AN^2 + NC^2 = AC^2$$

$$AN^2 + \left(\frac{a}{2} + ND\right)^2 = b^2$$

$$AN^2 + ND^2 + \frac{a^2}{4} + a(ND) = b^2$$

$$m^2 + \frac{a^2}{4} + a(ND) = b^2 \qquad \text{(from (1), } AN^2 + ND^2 = m^2\text{)}$$

$$m^2 + \frac{a^2}{4} + \left(m^2 + \frac{a^2}{4} - c^2\right) = b^2 \qquad \text{(setting value of } a(ND) \text{ from (2),)}$$

$$m^2 + \frac{a^2}{4} + m^2 + \frac{a^2}{4} - c^2 = b^2$$

$$2m^2 + \frac{a^2}{2} = b^2 + c^2$$

$$b^2 + c^2 = \frac{a^2}{2} + 2m^2$$

Proved

Apollonius Theorem for parallelogram: In a parallelogram, the sum of squares of diagonals is equal to twice the sum of the squares of its adjacent sides i.e. if a & b are lengths of two adjacent sides AB & BC respectively and d_1 & d_2 are the lengths of diagonals AC and BD respectively in a parallelogram $ABCD$ (as shown in fig-5) then

$$d_1^{\ 2} + d_2^{\ 2} = 2(a^2 + b^2)$$

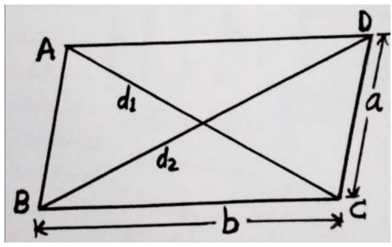

Fig-5: Parallelogram ABCD with adjacent sides a, b & diagonals d_1, d_2

Proof: Consider a parallelogram $ABCD$ having adjacent sides $AB = a$ & $BC = b$ and the diagonals $AC = d_1$ & $BD = d_2$. Let the diagonals AC and BD be bisecting each other at the point O (as shown in the fig-6) then we have

$$AO = OC = \frac{d_1}{2} \ \ \& \ BO = OD = \frac{d_2}{2}$$

In ΔABC, BO is median. Now, applying Apollonius theorem in this ΔABC as follows

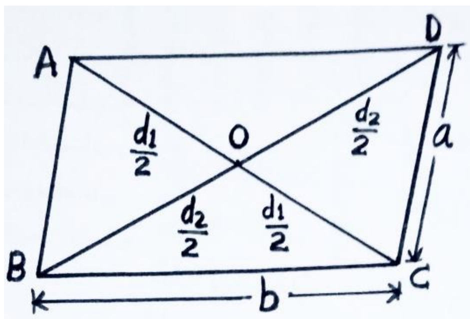

Fig-6: In parallelogram $ABCD$, $AO = OC = {d_1}/_2$ & $BO = OD = {d_2}/_2$

$$AB^2 + BC^2 = \frac{AC^2}{2} + 2(BO)^2$$

$$a^2 + b^2 = \frac{{d_1}^2}{2} + 2\left(\frac{d_2}{2}\right)^2$$

$$a^2 + b^2 = \frac{{d_1}^2}{2} + \frac{{d_2}^2}{2}$$

$$a^2 + b^2 = \frac{{d_1}^2 + {d_2}^2}{2}$$

$${d_1}^2 + {d_2}^2 = 2(a^2 + b^2) \qquad \textbf{Proved}$$

Special Case 1: A rhombus has all four sides equal hence setting $b = a$ in above formula of parallelogram, we get

$${d_1}^2 + {d_2}^2 = 2(a^2 + a^2)$$

$${d_1}^2 + {d_2}^2 = 4a^2$$

The above result can also be obtained by using Pythagoras theorem in a rhombus.

Special Case 2: A rectangle has both diagonals equal hence setting $d_1 = d_2 = d$ in above formula of parallelogram, we get

$$d^2 + d^2 = 2(a^2 + b^2)$$

$$2d^2 = 2(a^2 + b^2)$$

$$d^2 = a^2 + b^2$$

The above result is true for a rectangle by Pythagoras theorem.

Note: Above articles had been concluded & *illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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