Solid angle subtended by a rectangular right pyramid (solid/hollow) at its apex

(Application of HCR's Theory of Polygon)

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Introduction: Here we are to derive the formula for finding out the solid angle subtended by a rectangular right pyramid at its apex by using formula of solid angle subtended by a rectangular plane at any point lying on the perpendicular passing through its centre which has already been derived in **HCR's Theory of Polygon**. The solid angle subtended by a rectangular right pyramid will be derived in terms of apex angles α & β (i.e. angles between two pairs of consecutive lateral edges meeting at the apex of rectangular right pyramid).

Derivation: Let there be a right pyramid (solid or hollow) with apex point 'P' & base as a rectangle ABCD such that the angles between two pairs of consecutive lateral edges PA & PB and PB & PC are α and β respectively (as shown in the figure-1).

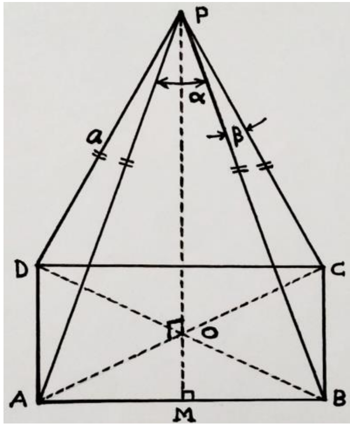

Fig-1: Perpendiculars PO & PM are dropped from apex P to the centre O of base ABCD & mid-point M of side AB in rectangular right pyramid

Now, drop the perpendiculars PO & PM on the rectangular base ABCD & side AB respectively & join the diagonals AC & BD of rectangular base ABCD (as shown by dotted lines in fig-1). Let a be the length of each of four equal lateral edges PA, PB, PC & PD of right pyramid.

In right ΔPMA, (see triangular face APB of right pyramid in fig-1)

$$\sin\angle APM = \frac{AM}{AP} \quad \Rightarrow \quad \sin\frac{\alpha}{2} = \frac{AM}{a}$$

$$AM = a\sin\frac{\alpha}{2}$$

$$\therefore AB = CD = 2AM = 2a\sin\frac{\alpha}{2}$$

Similarly, in isosceles ΔPBC , it can be proved by dropping a perpendicular from apex P to the side BC,

$$BC = AD = 2a\sin\frac{\beta}{2}$$

Using Pythagorean theorem in right ΔABC,

$$AC^2 = AB^2 + BC^2 = \left(2a\sin\frac{\alpha}{2}\right)^2 + \left(2a\sin\frac{\beta}{2}\right)^2 = 4a^2\left(\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2}\right)$$

$$AC = \sqrt{4a^2\left(\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2}\right)} = 2a\sqrt{\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2}}$$

$$AO = \frac{AC}{2} = \frac{2a\sqrt{\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2}}}{2} = a\sqrt{\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2}}$$

Using Pythagorean theorem in right ΔPOA (see above fig-1),

$$PA^2 = AO^2 + PO^2 \quad \Rightarrow \quad PO = \sqrt{PA^2 - AO^2}$$

$$PO = \sqrt{a^2 - \left(a\sqrt{\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2}}\right)^2}$$

$$= a\sqrt{1 - \sin^2\frac{\alpha}{2} - \sin^2\frac{\beta}{2}}$$

$$= a\sqrt{\cos^2\frac{\alpha}{2} - \sin^2\frac{\beta}{2}}$$

Now, the solid angle ($\omega_{pyramid}$) subtended by the rectangular right pyramid at its apex P will be equal to the solid angle ($\omega_{rectangle}$) subtended by rectangle ABCD of length and width l & b, at the apex P lying a normal height h from the centre 'O' which is given by the general formula of **HCR's Theory of Polygon** as follows

$$\boldsymbol{\omega_{rectangle} = 4\sin^{-1}\left(\frac{lb}{\sqrt{(l^2 + 4h^2)(b^2 + 4h^2)}}\right)}$$

Now, setting the value of normal height $h = PO$, length $l = AB$ & width $b = BC$ in the above general formula of solid angle, we get the solid angle subtended by the rectangular right pyramid at its apex

$$\omega_{pyramid} = 4\sin^{-1}\left(\frac{(AB)(BC)}{\sqrt{((AB)^2 + 4(PO)^2)((BC)^2 + 4(PO)^2)}}\right)$$

$$= 4\sin^{-1}\left(\frac{\left(2a\sin\frac{\alpha}{2}\right)\left(2a\sin\frac{\beta}{2}\right)}{\sqrt{\left(\left(2a\sin\frac{\alpha}{2}\right)^2 + 4\left(a\sqrt{\cos^2\frac{\alpha}{2} - \sin^2\frac{\beta}{2}}\right)^2\right)\left(\left(2a\sin\frac{\beta}{2}\right)^2 + 4\left(a\sqrt{\cos^2\frac{\alpha}{2} - \sin^2\frac{\beta}{2}}\right)^2\right)}}\right)$$

$$= 4\sin^{-1}\left(\frac{4a^2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\sqrt{(4a^2)^2\left(\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} - \sin^2\frac{\beta}{2}\right)\left(\sin^2\frac{\beta}{2} + \cos^2\frac{\alpha}{2} - \sin^2\frac{\beta}{2}\right)}}\right)$$

$$= 4\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\sqrt{\left(1 - \sin^2\frac{\beta}{2}\right)\left(\cos^2\frac{\alpha}{2}\right)}}\right)$$

$$= 4\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\sqrt{\left(\cos^2\frac{\beta}{2}\right)\left(\cos^2\frac{\alpha}{2}\right)}}\right)$$

$$= 4\sin^{-1}\left(\frac{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}\right)$$

$$= 4\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$$

Hence, the **solid angle (ω) subtended at the apex by any right pyramid with a rectangular base & the apex angles α & β (i.e. angles between two pairs of consecutive lateral edges meeting at apex)**, is given by the following formula

$$\boldsymbol{\omega = 4\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)}$$

$$\text{Where,}\quad \mathbf{0 < \alpha + \beta < \pi}$$

Important deduction: The solid angle subtended at the apex by a right pyramid with a square base & the apex angle α is obtained by substituting $\beta = \alpha$ in above general equation of solid angle, we get

$$\omega = 4\sin^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\alpha}{2}\right)$$

$$\boldsymbol{\omega = 4\sin^{-1}\left(\tan^2\frac{\alpha}{2}\right)}$$

$$\text{Where,}\quad \mathbf{0 < \alpha < \frac{\pi}{2}}$$

The above value of solid angle subtended at apex by a square right pyramid can also be proved by substituting $n = \text{number of sides of square base} = 4$ in general formula of solid angle subtended at apex by a regular n-gonal right pyramid (which has been derived in the paper '**solid angle subtended by a regular n-gonal right pyramid at its apex**' by the author), given as follows

$$\omega = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}\right)$$

$$= 2\pi - 2(4)\sin^{-1}\left(\cos\frac{\pi}{4}\sqrt{\tan^2\frac{\pi}{4} - \tan^2\frac{\alpha}{2}}\right) \qquad (\text{setting n} = 4 \text{ for square base})$$

$$= 2\pi - 8\sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{1 - \tan^2\frac{\alpha}{2}}\right)$$

$$= 2\pi - 4\sin^{-1}\left(2\frac{1}{\sqrt{2}}\sqrt{1 - \tan^2\frac{\alpha}{2}}\sqrt{1 - \left(\frac{1}{\sqrt{2}}\sqrt{1 - \tan^2\frac{\alpha}{2}}\right)^2}\right) \qquad \left(2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)\right)$$

$$= 2\pi - 4\sin^{-1}\left(2\frac{1}{\sqrt{2}}\sqrt{1 - \tan^2\frac{\alpha}{2}}\,\frac{1}{\sqrt{2}}\sqrt{1 + \tan^2\frac{\alpha}{2}}\right)$$

$$= 2\pi - 4\sin^{-1}\left(\sqrt{1 - \tan^4\frac{\alpha}{2}}\right)$$

$$= 2\pi - 4\cos^{-1}\left(\sqrt{1 - \left(\sqrt{1 - \tan^4\frac{\alpha}{2}}\right)^2}\right)$$

$$= 2\pi - 4\cos^{-1}\left(\sqrt{1 - 1 + \tan^4\frac{\alpha}{2}}\right)$$

$$= 2\pi - 4\cos^{-1}\left(\tan^2\frac{\alpha}{2}\right)$$

$$= 4\left(\frac{\pi}{2} - \cos^{-1}\left(\tan^2\frac{\alpha}{2}\right)\right)$$

$$= 4\left(\sin^{-1}\left(\tan^2\frac{\alpha}{2}\right)\right) \qquad \left(\text{Since, } \frac{\pi}{2} - \cos^{-1}x = \sin^{-1}x\right)$$

$$= 4\sin^{-1}\left(\tan^2\frac{\alpha}{2}\right) \qquad \textbf{Proved.}$$

Note: Above articles had been *derived & illustrated* by **Mr H.C. Rajpoot** (**M Tech, Production Engineering**)

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