$Mathematical\ Analysis\ of\ Rhombic\ Dodecahedron$

$(Application\ of\ HCR's\ Theory\ of\ Polygon)$

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Introduction: Here we are interested to mathematically analyse & derive analytic formula for a rhombic dodecahedron having 12 congruent faces each as a **rhombus**, 24 edges & 14 vertices out of which 6 identical vertices lie on a spherical surface with a certain radius while other 8 identical vertices do not lie on the same sphere. All 12 rhombic faces are at an equal normal distance from the centre of the rhombic dodecahedron. It is not vertex-transitive because 4 straight edges meet at 6 out of 14 vertices, while at rest 8 vertices, only 3 straight edges meet. It is dual of Cuboctahedron (an Archimedean solid) which is also called Catalan solid. This convex polyhedron can be constructed by joining 12 congruent elementary-right pyramids with rhombic base. By using '**HCR's Theory of Polygon**', we will derive the analytic formula which are very useful to compute the angles & diagonals of rhombic face, radii of circumscribed & inscribed spheres, surface area & volume of rhombic dodecahedron in terms of edge length a. (See fig-1, a rhombic dodecahedron inscribed in a sphere such that 6 identical vertices lie on sphere).

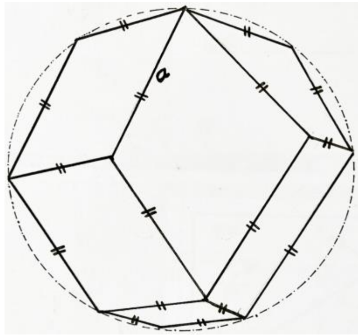

Figure 1: A rhombic dodecahedron with 12 congruent rhombic faces, 24 edges, and 14 vertices out of which 6 vertices lie on a sphere

Derivation of radius R of spherical surface circumscribing a given rhombic dodecahedron with edge length a: Consider a rhombic dodecahedron having 12 congruent faces each as a rhombus of side a (say rhombus ABCD as shown in fig-2). It is to be noted that only 6 identical vertices out of 14 vertices are lying on a spherical surface of radius R while other 8 identical vertices do not lie on this sphere of radius R (as shown in fig-1 above). It is worth noticing that major (larger) diagonals of four (congruent) rhombic faces, meeting at four common vertices, divide the great circle on the circumscribed sphere into four equal parts (each as a quarter great circle) thus each of four major diagonals say major diagonal AC of rhombic face ABCD exerts an angle of 90^0 at the centre O of spherical surface (as shown in fig-3 below). Now drop a perpendicular OM from centre O of spherical surface to the centre M of rhombic face ABCD. In right ΔAOC (see fig-3)

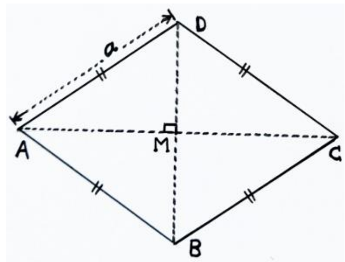

Figure 2: A rhombic face ABCD with each side a of polyhedron

$$\cos\frac{\pi}{4}=\frac{OM}{OC} \quad\Rightarrow\quad \boldsymbol{OM}=OC\cos\frac{\pi}{4}=R\cdot\frac{1}{\sqrt{2}}=\frac{R}{\sqrt{2}}$$

Similarly, (see fig. 3)

$$\boldsymbol{AM}=CM=OC\sin\frac{\pi}{4}=R\cdot\frac{1}{\sqrt{2}}=\frac{R}{\sqrt{2}}$$

In right ΔAMD (see fig-2 above), using Pythagorean theorem, we get

$$\boldsymbol{MD}=\sqrt{(AD)^2-(AM)^2}=\sqrt{(a)^2-\left(\frac{R}{\sqrt{2}}\right)^2}=\sqrt{a^2-\frac{R^2}{2}}$$

It is clear from above value of semi-minor diagonal MD of rhombus ABCD that

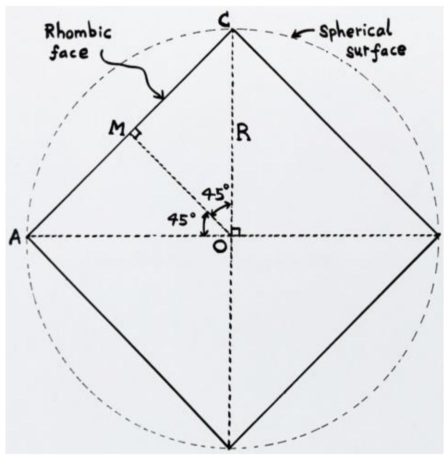

Figure 3: Four congruent rhombic faces are circumscribed by a sphere & $\perp$ to the plane of paper. Rhombic face ABCD, shown by its major diagonal AC with centre M, exerts an angle 90^0 at the centre O of sphere

$$a^2-\frac{R^2}{2}\geq 0 \quad \Rightarrow \quad \boldsymbol{R\leq a\sqrt{2}}$$

From '**HCR's Theory of Polygon'** (refer to the research paper for detailed derivation)**,** we know that the solid angle ω subtended by a right triangle say ΔQST, with perpendicular p & base b (as shown in fig-4), at a point say $P(0,0,h)$ lying on perpendicular at a height h from the right angled vertex S is given as

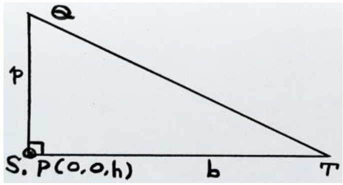

Figure 4: Point $\boldsymbol{P(0,0,h)}$ is $\perp$ to the plane of paper at height $\boldsymbol{h}$

$$\boldsymbol{\omega_{\Delta QST}=\cos^{-1}\left(\frac{h\left(p^2\sqrt{h^2+b^2}+b^2\sqrt{h^2+p^2}\right)}{h^2(p^2+b^2)+p^2b^2}\right)}$$

Now, the solid angle ($\omega_{\Delta AMD}$), subtended by right ΔAMD (of rhombic face ABCD) at the centre O (of spherical surface/rhombic dodecahedron) lying at a perpendicular distance $OM=R/\sqrt{2}$ from right angled vertex M (see fig-5), is obtained by substituting the normal height, $h=OM=R/\sqrt{2}$, perpendicular, $p=MD=\sqrt{a^2-\frac{R^2}{2}}$ & base, $b=AM=R/\sqrt{2}$ (as derived above) in the above generalized formula (from HCR's Theory of Polygon) of solid angle, is given as follows

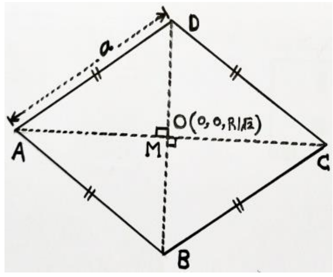

Figure 5: Centre $\boldsymbol{O(0,0,R/\sqrt{2})}$ is $\perp$ to the plane of paper at a normal height $\boldsymbol{R/\sqrt{2}}$ from right angled vertex M of $\boldsymbol{\Delta AMD}$

$$\omega_{\Delta AMD}=\cos^{-1}\left(\frac{\frac{R}{\sqrt{2}}\left(\left(\sqrt{a^2-\frac{R^2}{2}}\right)^2\sqrt{\left(\frac{R}{\sqrt{2}}\right)^2+\left(\frac{R}{\sqrt{2}}\right)^2}+\left(\frac{R}{\sqrt{2}}\right)^2\sqrt{\left(\frac{R}{\sqrt{2}}\right)^2+\left(\sqrt{a^2-\frac{R^2}{2}}\right)^2}\right)}{\left(\frac{R}{\sqrt{2}}\right)^2\left(\left(\sqrt{a^2-\frac{R^2}{2}}\right)^2+\left(\frac{R}{\sqrt{2}}\right)^2\right)+\left(\sqrt{a^2-\frac{R^2}{2}}\right)^2\left(\frac{R}{\sqrt{2}}\right)^2}\right)$$

$$=\cos^{-1}\left(\frac{\frac{R}{\sqrt{2}}\left(\left(a^2-\frac{R^2}{2}\right)\sqrt{\frac{R^2}{2}+\frac{R^2}{2}}+\frac{R^2}{2}\sqrt{\frac{R^2}{2}+a^2-\frac{R^2}{2}}\right)}{\frac{R^2}{2}\left(a^2-\frac{R^2}{2}+\frac{R^2}{2}\right)+\left(a^2-\frac{R^2}{2}\right)\frac{R^2}{2}}\right)$$

$$=\cos^{-1}\left(\frac{\frac{R}{\sqrt{2}}\left(R\left(a^2-\frac{R^2}{2}\right)+\frac{aR^2}{2}\right)}{\frac{a^2R^2}{2}+\frac{a^2R^2}{2}-\frac{R^4}{4}}\right)$$

$$=\cos^{-1}\left(\frac{\frac{R^2}{\sqrt{2}}\left(a^2-\frac{R^2}{2}+\frac{aR}{2}\right)}{R^2\left(a^2-\frac{R^2}{4}\right)}\right)$$

$$=\cos^{-1}\left(\frac{\frac{1}{\sqrt{2}}\left(\frac{2a^2+aR-R^2}{2}\right)}{\left(\frac{4a^2-R^2}{4}\right)}\right)$$

$$=\cos^{-1}\left(\frac{\sqrt{2}(2a^2+aR-R^2)}{4a^2-R^2}\right)$$

$$= \cos^{-1}\left(\frac{\sqrt{2}(R^2 - aR - 2a^2)}{R^2 - 4a^2}\right)$$

$$= \cos^{-1}\left(\frac{\sqrt{2}(R^2 - 2aR + aR - 2a^2)}{R^2 - (2a)^2}\right)$$

$$= \cos^{-1}\left(\frac{\sqrt{2}(R + a)(R - 2a)}{(R + 2a)(R - 2a)}\right)$$

$$\omega_{\Delta AMD} = \cos^{-1}\left(\frac{\sqrt{2}(R + a)}{R + 2a}\right)$$

The rhombic face ABCD is divided into four congruent right triangles AMB, AMD, CMB & CMD (as shown in above fig-2 or fig-5), therefore using symmetry, the solid angle subtended by the rhombic face ABCD at the centre O of spherical surface/rhombic dodecahedron is given as follows

$$\omega_{ABCD} = 4 \times (\text{Solid angle } \omega_{\Delta AMD} \text{ subtended by right } \Delta\text{AMD at the centre O })$$

$$\omega_{ABCD} = 4 \times \left(\cos^{-1}\left(\frac{\sqrt{2}(R + a)}{R + 2a}\right)\right)$$

$$\omega_{ABCD} = 4\cos^{-1}\left(\frac{\sqrt{2}(R + a)}{R + 2a}\right) \qquad \dots\dots(I)$$

But we know that solid angle, subtended by any closed surface at any point inside it, is always equal to 4π sr (this fact has also been mathematically proven by the author). The rhombic dodecahedron is a closed surface consisting of 12 congruent rhombic faces therefore using symmetry of the faces, the solid angle subtended at the centre O by each rhombic face say ABCD is given as follows

$$\omega_{ABCD} = \frac{\text{Total solid angle}}{\text{Number of congruent rhombic faces}} = \frac{4\pi}{12} = \frac{\pi}{3}$$

$$\omega_{ABCD} = \frac{\pi}{3} \qquad \dots\dots(II)$$

Now, equating the values of solid angle from (I) & (II) as follows

$$4\cos^{-1}\left(\frac{\sqrt{2}(R + a)}{R + 2a}\right) = \frac{\pi}{3}$$

$$\cos^{-1}\left(\frac{\sqrt{2}(R + a)}{R + 2a}\right) = \frac{\pi}{12}$$

$$\frac{\sqrt{2}(R + a)}{R + 2a} = \cos\frac{\pi}{12}$$

$$\frac{\sqrt{2}(R + a)}{R + 2a} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$4(R + a) = (\sqrt{3} + 1)(R + 2a)$$

$$4R + 4a = (\sqrt{3} + 1)R + 2(\sqrt{3} + 1)a$$

$$(4-\sqrt{3}-1)R = (2\sqrt{3}+2-4)a$$

$$(3-\sqrt{3})R = (2\sqrt{3}-2)a$$

$$R = \frac{(2\sqrt{3}-2)a}{(3-\sqrt{3})}$$

$$R = \frac{2(\sqrt{3}-1)a}{\sqrt{3}(\sqrt{3}-1)}$$

$$R = \frac{2a}{\sqrt{3}}$$

Therefore, the **radius (R) of spherical surface passing through 6 identical vertices of a given rhombic dodecahedron with edge length a** is given as follows

$$\boldsymbol{R = \frac{2a}{\sqrt{3}} \approx 1.154700538\, a} \qquad \dots\dots\dots\dots(1)$$

Diagonals & angles of rhombic face: Substituting the value of R in the relation (as derived above), the **length of major diagonal AC of rhombic face ABCD** of rhombic dodecahedron (as shown in above fig-5) is given as

$$AC = AM + MC = 2AM = 2\left(\frac{R}{\sqrt{2}}\right) = 2\left(\frac{\frac{2a}{\sqrt{3}}}{\sqrt{2}}\right) = 2a\sqrt{\frac{2}{3}}$$

$$\boldsymbol{\therefore \text{Major diagonal}, \ AC = 2a\sqrt{\frac{2}{3}} \approx 1.632993162\, a} \qquad \dots\dots\dots\dots(2)$$

Similarly, substituting the value of R in the relation (as derived above), the **length of minor diagonal BD of rhombic face ABCD** of rhombic dodecahedron (as shown in above fig-5) is given as

$$BD = BM + MD = 2MD = 2\left(\sqrt{a^2 - \frac{R^2}{2}}\right) = 2\left(\sqrt{a^2 - \frac{\left(\frac{2a}{\sqrt{3}}\right)^2}{2}}\right) = 2a\sqrt{1-\frac{2}{3}} = 2a\sqrt{\frac{1}{3}}$$

$$\boldsymbol{\therefore \text{Minor diagonal}, \ BD = \frac{2a}{\sqrt{3}} \approx 1.154700538\, a} \qquad \dots\dots\dots\dots(3)$$

From above eq(2) & (3), it is interesting to note that the **major diagonal (AC) is $\sqrt{2}$ times the minor diagonal (BD) of rhombic face (ABCD) of any rhombic dodecahedron**.

In right ΔAMD (see above fig-2 or fig-5),

$$\tan\angle ADM = \frac{AM}{MD} = \frac{2AM}{2MD} = \frac{AC}{BD} = \frac{2a\sqrt{\frac{2}{3}}}{\frac{2a}{\sqrt{3}}} = \sqrt{2}$$

$$\angle ADM = \tan^{-1}\sqrt{2}$$

$$\Rightarrow \angle ADC = 2\angle ADM = 2\tan^{-1}\sqrt{2} \qquad (\text{since, minor diagonal BD bisects } \angle ADC)$$

From property of supplementary angles in rhombus ABCD (see above fig-5), we have

$$\angle BAD + \angle ADC = \pi$$

$$\angle BAD = \pi - \angle ADC$$

$$\angle BAD = \pi - 2\tan^{-1}\sqrt{2}$$

$$\angle BAD = 2\left(\frac{\pi}{2} - \tan^{-1}\sqrt{2}\right)$$

$$\angle BAD = 2\left(\cot^{-1}\sqrt{2}\right)$$

Thus, the **acute and obtuse angles α & β respectively of rhombic face ABCD** of rhombic dodecahedron (as shown in above fig-5) are given as

$$\boldsymbol{\alpha = 2\cot^{-1}\sqrt{2} \approx 70^0 31' 43.6''} \ \ \& \ \ \boldsymbol{\beta = 2\tan^{-1}\sqrt{2} \approx 109^0 28' 16.4''} \qquad \dots\dots\dots\dots(4)$$

Radius (R_i) of sphere inscribed by rhombic dodecahedron or normal distance (H) of each rhombic face from the centre of rhombic dodecahedron: From above fig-3, it is clear that the radius R_i of the sphere touching all 12 congruent rhombic faces i.e. radius R_i of sphere inscribed by the rhombic dodecahedron is equal to the normal distance OM from the centre O to the centre M of rhombic face ABCD (see fig-3 above). Thus the normal distance OM $(= H)$ of each rhombic face from the centre O is given as

$$OM = \frac{R}{\sqrt{2}} = \frac{\frac{2a}{\sqrt{3}}}{\sqrt{2}} = a\sqrt{\frac{2}{3}} \qquad (\text{as derived above from fig. 3})$$

$$\boldsymbol{\therefore \text{Inscribed radius } (R_i) = \text{Normal distance (H)} = a\sqrt{\frac{2}{3}} \approx 0.81649658\, a} \qquad \dots\dots\dots\dots(5)$$

Surface Area (A_s) of rhombic dodecahedron: The surface of a rhombic dodecahedron consists of 12 congruent flat faces each as a rhombus of side a therefore its total surface area is given as follows

$$A_s = 12 \times (\text{Area of rhombic face ABCD})$$

$$= 12 \times (4 \times \text{Area of right } \Delta\text{AMD}) \qquad (\text{See symmetry in above fig. 2})$$

$$= 48(\text{Area of right } \Delta\text{AMD})$$

$$= 48\left(\frac{1}{2}(AM)(BD)\right)$$

$$= 48\left(\frac{1}{2}\left(a\sqrt{\frac{2}{3}}\right)\left(\frac{a}{\sqrt{3}}\right)\right) \qquad (\text{Setting values of semi diagonals AM \& } BD)$$

$$= 48\left(\frac{a^2\sqrt{2}}{6}\right)$$

$$= 8a^2\sqrt{2}$$

$$\therefore \textbf{Surface area of rhombic dodecahedron}, \ A_s = 8a^2\sqrt{2} \approx 11.3137085\ a^2$$

Volume (V) of rhombic dodecahedron: A rhombic dodecahedron has 12 congruent faces each as a rhombus of side a thus it can constructed by joining 12 congruent elementary right pyramids each with a rhombic base of side a & normal height H such that apex-points of all pyramids coincide at the centre O of rhombic dodecahedron (as shown in fig-6). Thus the volume of rhombic dodecahedron is given as

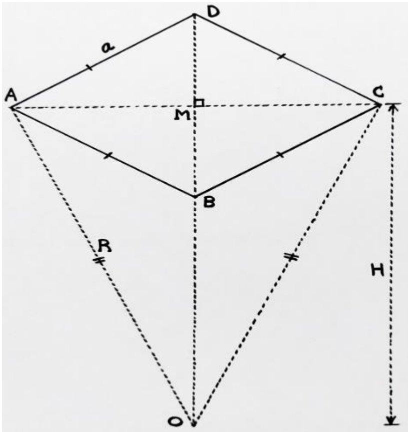

Figure 6: An elementary right pyramid OABCD is obtained by joining all four vertices A, B, C & D of rhombic face ABCD to the centre O of polyhedron

$$V = 12 \times (\text{Volume of right pyramid OABCD})$$

$$= 12\left(\frac{1}{3}(\text{Area of rhombus ABCD})(\text{Normal height})\right)$$

$$= 12\left(\frac{1}{3}(4 \times \text{Area of right } \Delta\text{AMD})(H)\right)$$

$$= 16(\text{Area of right } \Delta\text{AMD})(H)$$

$$= 16\left(\frac{1}{2}(AM)(BD)\right)\left(a\sqrt{\frac{2}{3}}\right) \qquad (\text{Setting value of normal height } H)$$

$$= 16\left(\frac{1}{2}\left(a\sqrt{\frac{2}{3}}\right)\left(\frac{a}{\sqrt{3}}\right)\right)\left(a\sqrt{\frac{2}{3}}\right) \qquad (\text{Setting values of semi diagonals AM } \& \ BD)$$

$$= 16\left(\frac{a^3}{3\sqrt{3}}\right)$$

$$= \frac{16a^3}{3\sqrt{3}}$$

$$\therefore \textbf{Volume of rhombic dodecahedron}, \ V = \frac{16a^3}{3\sqrt{3}} \approx 3.079201436\ a^3$$

Mean radius (R_m) of rhombic dodecahedron: It is the radius of the sphere having a volume equal to that of a given rhombic dodecahedron with edge length a. It is computed as follows

$$\text{volume of sphere with mean radius } R_m = \text{volume of rhombic dodecahedron with edge length } a$$

$$\frac{4}{3}\pi(R_m)^3 = \frac{16a^3}{3\sqrt{3}}$$

$$(R_m)^3 = \frac{4a^3}{\pi\sqrt{3}}$$

$$R_m = \left(\frac{4a^3}{\pi\sqrt{3}}\right)^{1/3}$$

$$R_m = a\left(\frac{4}{\pi\sqrt{3}}\right)^{1/3}$$

$$\therefore \textbf{ Mean radius, } \boldsymbol{R_m = a\left(\frac{4}{\pi\sqrt{3}}\right)^{1/3} \approx 0.902505444\, a} \qquad \ldots\ldots\ldots\ldots(6)$$

Radius (R_{md}) of midsphere (intersphere) of rhombic dodecahedron: It is the radius of the sphere touching each of 24 equal edges at a single point. Consider a rhombic face ABCD whose sides are touching the midsphere of radius R_{md} at four distinct points E, F, G & I. Draw a circle with centre at point M, passing through points of tangency E, F, G & I which are joined to the centre M by straight lines (as shown in fig-7). This tangent circle with centre M lies on the surface of midsphere. Now, join the points M, E, F, G & I to the centre O of rhombic dodecahedron. (see fig-7).

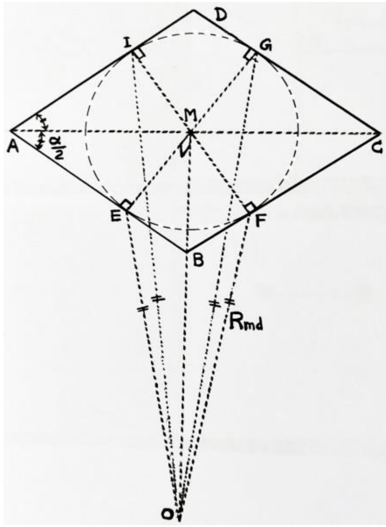

Figure 7: A Circle inscribed by rhombic face ABCD lies on the midsphere with radius R_{md} of rhombic dodecahedron Where, $OE = OF = OG = OI = R_{md}$

In right ΔAEM (see fig-7),

$$\sin\angle EAM = \frac{EM}{AM} \quad \Rightarrow \quad EM = AM\sin\angle EAM$$

$$EM = a\sqrt{\frac{2}{3}}\sin\frac{\alpha}{2} \qquad \left(\text{since, } AM = a\sqrt{\frac{2}{3}}\ \ \&\ \angle EAM = \frac{\angle BAD}{2} = \frac{\alpha}{2}\right)$$

$$EM = a\sqrt{\frac{2}{3}}\sin\left(\frac{2\cot^{-1}\sqrt{2}}{2}\right) \qquad (\text{Setting the value of } \alpha)$$

$$EM = a\sqrt{\frac{2}{3}}\sin\left(\cot^{-1}\sqrt{2}\right)$$

$$EM = a\sqrt{\frac{2}{3}}\sin\left(\sin^{-1}\frac{1}{\sqrt{3}}\right) \qquad \left(\text{since, } \cot^{-1}x = \sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)$$

$$EM = a\sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{3}}\right) = \frac{a\sqrt{2}}{3}$$

In right ΔOME (see above fig-7), using Pythagorean theorem, we get

$$OE = \sqrt{(OM)^2 + (EM)^2}$$

$$R_{md} = \sqrt{\left(a\sqrt{\frac{2}{3}}\right)^2 + \left(\frac{a\sqrt{2}}{3}\right)^2} \qquad (\text{Setting the values of OM \& EM as derived above})$$

$$= \sqrt{\frac{2a^2}{3} + \frac{2a^2}{9}}$$

$$= \sqrt{\frac{8a^2}{9}} = \frac{2a\sqrt{2}}{3}$$

$$\therefore \textbf{Radius of midsphere}, \ R_{md} = \frac{2a\sqrt{2}}{3} \approx 0.942809041\, a \qquad \dots\dots\dots\dots(7)$$

It is very interesting to note that for finite value of edge length $a \Rightarrow \boldsymbol{R_i < R_m < R_{md} < R}$

We know that a rhombic dodecahedron has total 14 vertices which are of two types. It has 8 identical & diagonally opposite vertices at each of which $n = 3$ no. of edges meet together & do not lie on a sphere of radius R. Rest 6 are identical & diagonally opposite vertices at each of which $n = 4$ no. of edges meet together and lie on a sphere of radius R. Thus we would analyse these two cases to compute solid angle subtended by the rhombic dodecahedron at its two dissimilar vertices A & B by assuming that the eye of the observer is located at any of two dissimilar vertices & directed (focused) straight to the centre of rhombic dodecahedron (as shown in fig-8 below). In order to distinguish these two types of vertices of a rhombic dodecahedron, let's use following symbols

$\boldsymbol{V\{3\}}$ represents a vertex of rhombic dodecahedron at which three edges meet together (Ex. vertex B)

$\boldsymbol{V\{4\}}$ represents a vertex of rhombic dodecahedron at which four edges meet together (Ex. vertex A).

Thus let's analyse both the cases for vertices $V\{3\}$ & $V\{4\}$ as follows

Solid angles subtended by rhombic dodecahedron at its vertices $V\{4\}$ & $V\{3\}$ (i.e. vertices A & B):

From '**HCR's Theory of Polygon'**, the **solid angle** (ω)**, subtended at the vertex (apex point) by a right pyramid with a regular n-gonal base & an angle α between any two consecutive lateral edges meeting at the same vertex,** is mathematically given by the standard (generalized) formula as follows

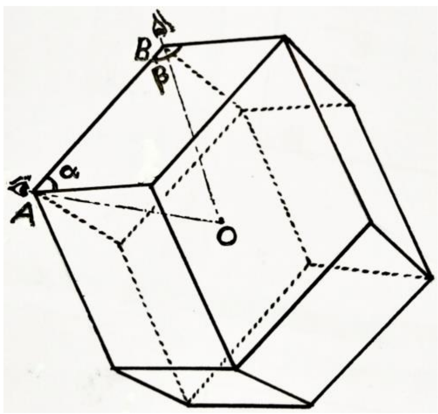

Figure 8: The eye of the observer is located at two different types of vertices $V\{4\}$ & $V\{3\}$ i.e. vertices A & B respectively

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}\right)} \qquad \forall\ n \in N\ \&\ n \geq 3$$

We know that 4 (straight) edges are meeting at the vertex $V\{4\}$ i.e. vertex A of rhombic dodecahedron such that the angle between any two consecutive edges is α (as shown in fig-8). It is very interesting to know that when an observer puts his one eye very close theoretically at the vertex A then the rhombic dodecahedron looks like a right pyramid with its apex at the eye (i.e. vertex A) where its four lateral edges meet together. Therefore the generalised formula (as given above from "HCR's Theory of Polygon") can be equally applied for finding the solid angle by rhombic dodecahedron at its vertex A (i.e. type $V\{4\}$). Hence substituting the corresponding values in above generalized formula as follows

$n = \text{number of edges meeting at vertex A} = 4,$
$\alpha = \text{angle between any two consecutive edges meeting at vertex A} = 2\cot^{-1}\sqrt{2}$ (as derived above)

We get the solid angle $\omega\{4\}$ subtended by the rhombic dodecahedron at vertex $V\{4\}$ (i.e. vertex A) as follows

$$\omega\{4\} = 2\pi - 2(4)\sin^{-1}\left(\cos\frac{\pi}{4}\sqrt{\tan^2\frac{\pi}{4} - \tan^2\left(\frac{2\cot^{-1}\sqrt{2}}{2}\right)}\right)$$

$$\omega\{4\} = 2\pi - 8\sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{1^2 - \left(\tan\left(\cot^{-1}\sqrt{2}\right)\right)^2}\right)$$

$$\omega\{4\} = 2\pi - 8\sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{1 - \left(\tan\left(\tan^{-1}\frac{1}{\sqrt{2}}\right)\right)^2}\right)$$

$$\omega\{4\} = 2\pi - 8\sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2}\right)$$

$$\omega\{4\} = 2\pi - 8\sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{1}{2}}\right)$$

$$\omega\{4\} = 2\pi - 8\sin^{-1}\left(\frac{1}{2}\right)$$

$$\omega\{4\} = 2\pi - 8\left(\frac{\pi}{6}\right) = 2\pi - \frac{4\pi}{3} = \frac{2\pi}{3}$$

Similarly, it can be noted that 3 (straight) edges of equal length are meeting at the vertex $V\{3\}$ i.e. vertex B of rhombic dodecahedron such that the angle between any two consecutive edges is β (as shown in above fig-8). It is very interesting to know that when an observer puts his one eye very close theoretically at the vertex B then the rhombic dodecahedron looks like a right pyramid with its apex at the eye (i.e. vertex B) where its three lateral edges meet together such that angle between any two consecutive edge is $2\tan^{-1}\sqrt{2}$.

Hence substituting the corresponding values in above HCR's generalized formula as follows

$n =$ number of edges meeting at vertex B $= 3$,

$\alpha =$ angle between any two consecutive edges meeting at vertex B $= 2\tan^{-1}\sqrt{2}$ (as derived above)

We get the solid angle $\omega\{3\}$ subtended by the rhombic dodecahedron at vertex $V\{3\}$ (i.e. vertex B) as follows

$$\omega\{3\} = 2\pi - 2(3)\sin^{-1}\left(\cos\frac{\pi}{3}\sqrt{\tan^2\frac{\pi}{3} - \tan^2\left(\frac{2\tan^{-1}\sqrt{2}}{2}\right)}\right)$$

$$\omega\{3\} = 2\pi - 6\sin^{-1}\left(\frac{1}{2}\sqrt{\left(\sqrt{3}\right)^2 - \left(\tan\left(\tan^{-1}\sqrt{2}\right)\right)^2}\right)$$

$$\omega\{3\} = 2\pi - 6\sin^{-1}\left(\frac{1}{2}\sqrt{3 - \left(\sqrt{2}\right)^2}\right)$$

$$\omega\{3\} = 2\pi - 6\sin^{-1}\left(\frac{1}{2}\right)$$

$$\omega\{3\} = 2\pi - 6\left(\frac{\pi}{6}\right) = 2\pi - \pi = \pi$$

Hence, the **solid angles $\boldsymbol{\omega\{3\}}$ & $\boldsymbol{\omega\{4\}}$ subtended by a rhombic dodecahedron at its vertices $\boldsymbol{V\{3\}}$ (where 3 edges meet)** & **$\boldsymbol{V\{4\}}$ (where 4 edges meet)** respectively are given as follows

$$\boldsymbol{\omega\{3\} = \pi\ \mathrm{sr} \quad \& \quad \omega\{4\} = \frac{2\pi}{3}\ \mathrm{sr}} \qquad \ldots\ldots\ldots\ldots.\ \mathbf{(8)}$$

From above values of solid angles $\omega\{3\} > \omega\{4\}$, it is very interesting to note that a rhombic dodecahedron appears larger when seen from vertex $V\{3\}$ (where 3 edges meet) as compared to when seen from vertex $V\{4\}$ (where 4 edges meet)

Dihedral angle between any two adjacent rhombic faces of a rhombic dodecahedron: We know that there are 12 congruent rhombic faces such that each two adjacent faces are inclined with another at an equal angle called dihedral angle. Consider two adjacent rhombic faces ABCD & ABEF with a common edge AB which are inclined at an angle θ. Drop the perpendiculars MP & NP from the centres M, N of faces to the common edge AB & join the points M, N & P to the centre O of polyhedron by dotted straight lines (see fig-9)

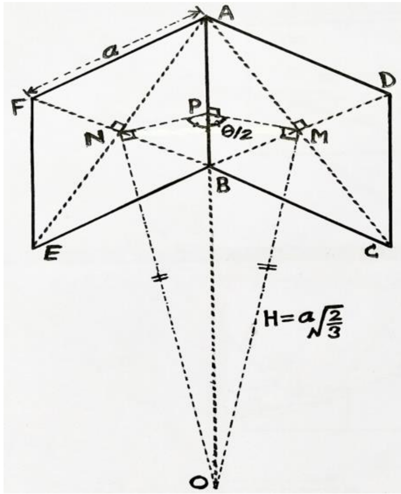

Figure 9: Dihedral angle $\angle MPN = \theta$ between two adjacent rhombic faces ABCD & ABEF is bisected by the line OP

In right ΔAMB (see fig-9), the length of perpendicular MP is given by generalized formula from 'HCR's derivations of some important formula in 2D-Goemetry' as follows

$$MP = \frac{(\text{Base})(\text{Perpendicular})}{\text{Hypotenuse}}$$

$$MP = \frac{(AM)(BM)}{AB}$$

$$MP = \frac{\left(a\sqrt{\frac{2}{3}}\right)\left(\frac{a}{\sqrt{3}}\right)}{a} \qquad \text{(Setting the values of AM \& BM)}$$

$$MP = \frac{\frac{a^2\sqrt{2}}{3}}{a} = \frac{a\sqrt{2}}{3}$$

In right ΔOMP (see above fig-9),

$$\tan\angle OPM = \frac{OM}{MP}$$

$$\tan\frac{\theta}{2} = \frac{a\sqrt{\frac{2}{3}}}{\frac{a\sqrt{2}}{3}} \qquad \text{(Setting the values of OM \& MP)}$$

$$\tan\frac{\theta}{2} = \sqrt{3} \Rightarrow \frac{\theta}{2} = \tan^{-1}\sqrt{3} \Rightarrow \frac{\theta}{2} = \frac{\pi}{3} \Rightarrow \theta = \frac{2\pi}{3}$$

Hence, the **dihedral angle θ between any two adjacent rhombic faces of a rhombic dodecahedron** is given as follows

$$\textbf{Dihedral angle, } \boldsymbol{\theta = \frac{2\pi}{3} = 120^o} \qquad \ldots\ldots\ldots\ldots(9)$$

Construction of a solid rhombic dodecahedron: In order to construct a solid rhombic dodecahedron having 12 congruent faces each as a rhombus of side a

1: Construct all its 12 congruent elementary right pyramids with rhombic base of side a & normal height H given as (See above fig-6)

$$\text{Normal height}, H = a\sqrt{\frac{2}{3}} \quad \& \text{ slant height} = \frac{2a}{\sqrt{3}}$$

2: Bond by joining all these 12 elementary right pyramids by overlapping their lateral faces & keeping their apex points coincident with each other such that 12 rhombic bases coincide one another at their edges. Thus, a solid rhombic dodecahedron is obtained.

Summary: Let there be any rhombic dodecahedron having 12 congruent faces each as a rhombus of side a, 24 edges & 14 vertices then all its important parameters are determined as tabulated below

Acute & obtuse angles $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$, major & minor diagonals $\boldsymbol{d_1}$ & $\boldsymbol{d_2}$ of rhombic face	$\alpha = 2\cot^{-1}\sqrt{2} \approx 70.53^0$ & $\beta = 2\tan^{-1}\sqrt{2} \approx 109.47^0$ $d_1 = 2a\sqrt{\frac{2}{3}} \approx 1.632993162\ a$ & $d_2 = \frac{2a}{\sqrt{3}} \approx 1.154700538\ a$
Radius $(\boldsymbol{R_i})$ of inscribed sphere or normal distance $(\boldsymbol{H})$ of each rhombic face from centre of polyhedron	$R_i = H = a\sqrt{\frac{2}{3}} \approx 0.81649658$
Mean radius $(\boldsymbol{R_m})$ or radius of sphere having volume equal to rhombic dodecahedron	$R_m = a\left(\frac{4}{\pi\sqrt{3}}\right)^{1/3} \approx 0.902505444\ a$
Radius $(\boldsymbol{R_{md}})$ of midsphere touching all the edges	$R_{md} = \frac{2a\sqrt{2}}{3} \approx 0.942809041\ a$
Radius$(\boldsymbol{R})$ of circumscribed sphere passing through 6 identical vertices out of total 14 vertices	$R = \frac{2a}{\sqrt{3}} \approx 1.154700538\ a$
Surface area $(\boldsymbol{A_s})$	$A_s = 8a^2\sqrt{2} \approx 11.3137085\ a^2$
Volume $(\boldsymbol{V})$	$V = \frac{16a^3}{3\sqrt{3}} \approx 3.079201436\ a^3$
Solid angles $\boldsymbol{\omega\{3\}}$ & $\boldsymbol{\omega\{4\}}$ subtended at vertices $\mathbf{V\{3\}}$ & $\mathbf{V\{4\}}$	$\omega\{3\} = \pi$ sr & $\omega\{4\} = \frac{2\pi}{3}$ sr
Dihedral angle $\boldsymbol{\theta}$ between any two adjacent rhombic faces	$\theta = \frac{2\pi}{3} = 120^o$

Note: Above articles had been *derived & illustrated* by **Mr H.C. Rajpoot** (**M Tech, Production Engineering**)

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