Mathematical Analysis of Truncated Rhombic Dodecahedron

(*Application of HCR's Theory of Polygon*)

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Introduction: A truncated rhombic dodecahedron is a non-uniform convex polyhedron which has 12 congruent rectangular faces, 6 congruent square faces & 8 congruent equilateral-triangular faces, 48 edges (24 small & 24 large edges) & 24 identical vertices (at each of which two rectangular, one square & one triangular faces meet) lying on a spherical surface of certain radius. It is derived by truncating all 14 vertices of a rhombic dodecahedron at the points of tangency (where an inscribed circle touches four sides of rhombic face) such that its every rhombic face is changed into a rectangular face which has its length $\sqrt{2}$ times the width and all its 24 edges are changed into 24 new identical vertices (as shown in fig-1). This truncated rhombic dodecahedron looks very similar to a rhombicuboctahedron but a truncated rhombic dodecahedron has 12 rectangular faces instead of square faces. The number of faces, edges & vertices of a truncated rhombic dodecahedron generated by truncating all the vertices of parent polyhedron (i.e. rhombic dodecahedron) are obtained as follows

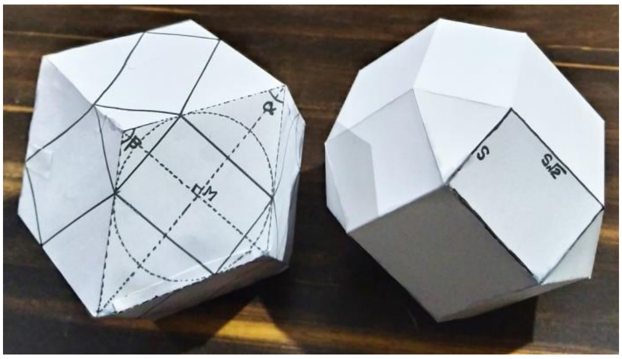

Figure 1: A rhombic dodecahedron (left) is marked to be truncated at each of its 14 vertices to generate 12 rectangular faces, 6 square faces & 8 equilateral triangular faces to form a truncated rhombic dodecahedron (right) with edges s & $s\sqrt{2}$

$$\text{Number of new rectangular faces} = \text{number of faces in parent solid (rhombic dodecahedron)} = 12$$

$$\text{Number of new square faces} = \text{number of vertices where 4 edges meet in parent solid} = 6$$

$$\text{Number of new triangular faces} = \text{number of vertices where 3 edges meet in parent solid} = 8$$

$$\text{Number of new edges} = (\text{no. of edges in rhombus})(\text{no. of rhombic faces in parent solid}) = 4 \cdot 12 = 48$$

$$\text{Number of new vertices} = \text{number of edges in parent solid} = 24$$

Since a truncated rhombic dodecahedron is derived from a rhombic dodecahedron hence it becomes much easier and simpler to mathematically analyse & derive analytic formula for a truncated rhombic dodecahedron by using the mathematical relations and formula derived for a rhombic dodecahedron (refer to '**Mathematical analysis of rhombic dodecahedron**' for derivations/formula). For this we will consider a parent rhombic dodecahedron of edge length a (which is truncated at the points of tangency of inscribed circle & rhombic face as shown in fig-1 above) to derive a truncated rhombic dodecahedron of small & large edge lengths $\& s\sqrt{2}$ respectively. We will establish a mathematical relation between edge length a of parent polyhedron (i.e. rhombic dodecahedron) & small edge length s of truncated rhombic dodecahedron. Thus we will derive the formula to analytically compute the radius of circumscribed sphere passing through all 24 identical vertices, normal distances of rectangular, square & equilateral triangular faces from the centre of polyhedron, surface area, volume, solid angles subtended by rectangular, square & equilateral triangular faces at the centre of polyhedron by using '**HCR's Theory of Polygon**', dihedral angle between each two faces meeting at any of 24 identical vertices, solid angle subtended by truncated rhombic dodecahedron at any of its 24 identical vertices.

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Derivation of radius R of circumscribed sphere i.e. passing through all 24 identical vertices of a truncated rhombic dodecahedron: Consider a rhombic dodecahedron having 12 congruent faces each as a rhombus of side a. We know that the midsphere with radius R_{md} touches all 24 edges of rhombic dodecahedron i.e. midsphere touches all four sides of each rhombic face. Consider a rhombic face ABCD touching the midsphere at four distinct points (of tangency) E, F, G & I on the sides AB, BC, CD & AD respectively. Circle passing through the points of tangency E, F, G & I is inscribed by the rhombus ABCD & lies on the surface of midsphere of rhombic dodecahedron. Join these four points of tangency E, F, G & I to get a rectangular face EFGI (as shown in fig-2). Thus we mark a rectangle on each of 12 congruent rhombic faces by joining the points of tangency of the edges & the midsphere and then truncate the rhombic dodecahedron from each of its 14 vertices at the points of tangency to get a truncated rhombic dodecahedron (as shown in above fig-1). From formula derived in 'Mathematical analysis of rhombic dodecahedron', the angles α & β, the lengths of semi major & semi minor diagonals AM & BM of rhombic face ABCD , and the radius R_{md} of midsphere of rhombic dodecahedron with edge length a , are given as

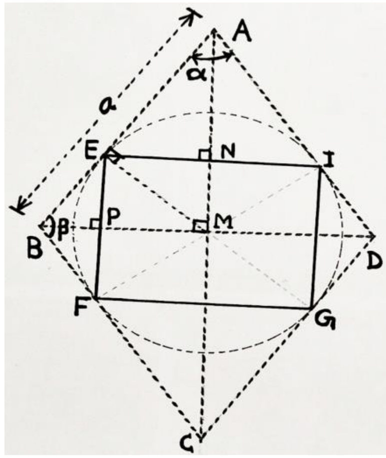

Figure 2: A rhombic face ABCD is changed into a rectangular face EFGI by truncating a rhombic dodecahedron from its vertices at points of tangency E, F, G & I of edges & the midsphere.

$$\alpha = 2\cot^{-1}\sqrt{2},\ \ \beta = 2\tan^{-1}\sqrt{2}\ ,\ \ AM = a\sqrt{\frac{2}{3}}\ ,\ \ BM = \frac{a}{\sqrt{3}}\ ,\ \ R_{md} = \frac{2a\sqrt{2}}{3}$$

In right ΔAEM (see fig-2)

$$\cos\frac{\alpha}{2} = \frac{AE}{AM} \quad \Rightarrow \quad AE = AM\cos\frac{\alpha}{2} = a\sqrt{\frac{2}{3}}\cos\left(\frac{2\cot^{-1}\sqrt{2}}{2}\right) = a\sqrt{\frac{2}{3}}\cos\left(\cos^{-1}\sqrt{\frac{2}{3}}\right) = a\sqrt{\frac{2}{3}}\left(\sqrt{\frac{2}{3}}\right) = \frac{2a}{3}$$

$$\Rightarrow \quad BE = AB - AE = a - \frac{2a}{3} = \frac{a}{3}$$

In right ΔANE (see fig-2)

$$\sin\frac{\alpha}{2} = \frac{EN}{AE} \quad \Rightarrow \quad \boldsymbol{EN} = AE\sin\frac{\alpha}{2} = \frac{2a}{3}\sin\left(\frac{2\cot^{-1}\sqrt{2}}{2}\right) = \frac{2a}{3}\sin\left(\sin^{-1}\frac{1}{\sqrt{3}}\right) = \frac{2a}{3}\left(\frac{1}{\sqrt{3}}\right) = \frac{2a}{3\sqrt{3}}$$

$$\Rightarrow \quad \boldsymbol{EI} = 2EN = 2\left(\frac{2a}{3\sqrt{3}}\right) = \frac{4a}{3\sqrt{3}} \qquad \ldots\ldots(\mathrm{I})$$

Similarly, in right ΔBPE (see fig-2 above)

$$\sin\frac{\beta}{2} = \frac{EP}{BE} \quad \Rightarrow \quad \boldsymbol{EP} = BE\sin\frac{\beta}{2} = \frac{a}{3}\sin\left(\frac{2\tan^{-1}\sqrt{2}}{2}\right) = \frac{a}{3}\sin\left(\sin^{-1}\sqrt{\frac{2}{3}}\right) = \frac{a}{3}\left(\sqrt{\frac{2}{3}}\right) = \frac{a}{3}\sqrt{\frac{2}{3}}$$

$$\Rightarrow \quad \boldsymbol{EF} = 2EP = 2\left(\frac{a}{3}\sqrt{\frac{2}{3}}\right) = \frac{2a}{3}\sqrt{\frac{2}{3}} \qquad \ldots\ldots(\mathrm{II})$$

Now, diving the eq. (I) by eq.(II) as follows

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$$\frac{EI}{EF}=\frac{\frac{4a}{3\sqrt{3}}}{\frac{2a}{3}\sqrt{\frac{2}{3}}}=\sqrt{2} \quad \Rightarrow EI=EF\sqrt{2} \ \Rightarrow \textbf{Length of rectangular face EFGI} = \sqrt{2}\times \textbf{Width}$$

If the small side or width EF of the rectangular face EFGI is s i.e. $EF=s$ then the large side or length EI of the rectangular face EFGI is $=EF\sqrt{2}=s\sqrt{2}$. Thus, two unequal edges i.e. length and width of each of 12 congruent rectangular faces of the truncated rhombic dodecahedron are $\& s\sqrt{2}$. Now, substituting $EF=s$ in eq.(2) as follows

$$EF=s=\frac{2a}{3}\sqrt{\frac{2}{3}} \qquad \Rightarrow \qquad \boldsymbol{a}=\frac{\boldsymbol{3s}}{\boldsymbol{2}}\sqrt{\frac{\boldsymbol{3}}{\boldsymbol{2}}} \qquad \ldots\ldots\ldots\ldots(III)$$

Now, the **radius R of the circumscribed sphere i.e. passing through all 24 identical vertices of the truncated rhombic dodecahedron with unequal edge lengths s & $s\sqrt{2}$** , is given as follows

$R=$ Radius R_{md} of midsphere of parent rhombic dodecahedron with edge length a

$$R=\frac{2a\sqrt{2}}{3} \qquad \text{(from formula derived for a rhombic dodecahedron)}$$

$$R=\frac{2\left(\frac{3s}{2}\sqrt{\frac{3}{2}}\right)\sqrt{2}}{3}=s\sqrt{3} \qquad \text{(Substituting value of } a \text{ in terms of } s \text{ from eq. (III))}$$

$$\boldsymbol{R=s\sqrt{3}\approx 1.732050808\ s} \qquad \ldots\ldots\ldots\ldots.(1)$$

Normal distances H_R, H_S & H_T of rectangular, square & equilateral triangular faces from the centre of a truncated rhombic dodecahedron: Consider a rectangular face EFGI of length and width s & $s\sqrt{2}$ which is at a normal distance $OM=H_R$ from the centre O of truncated rhombic dodecahedron. Join its vertices E, F, G & I & centre M to the centre O (as shown in fig-3)

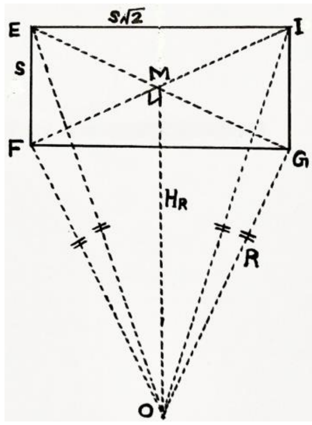

Figure 3: Rectangular face EFGI is at a normal distance H_R from centre O of polyhedron. $OE=OF=OG=OI=R$

In right ΔOMG (see fig-3), using Pythagorean theorem, we get

$$OM=\sqrt{(OG)^2-(MG)^2}=\sqrt{(R)^2-\left(\frac{EG}{2}\right)^2}=\sqrt{\left(s\sqrt{3}\right)^2-\left(\frac{\sqrt{(s)^2+\left(s\sqrt{2}\right)^2}}{2}\right)^2}$$

$$\boldsymbol{H_R}=\sqrt{3s^2-\frac{3s^2}{4}}=\sqrt{\frac{9s^2}{4}}=\frac{3s}{2}$$

Consider a square face EKLI of side $s\sqrt{2}$ which is at a normal distance $OQ=H_S$ from the centre O of truncated rhombic dodecahedron. Join its vertices E, K, L & I & centre Q to the centre O (as shown in fig-4 below)

In right ΔOQL (see fig-4 below), using Pythagorean theorem, we get

$$OQ=\sqrt{(OL)^2-(QL)^2}$$

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$$H_S = \sqrt{(R)^2 - \left(\frac{EL}{2}\right)^2}$$

$$H_S = \sqrt{\left(s\sqrt{3}\right)^2 - \left(\frac{\sqrt{\left(s\sqrt{2}\right)^2 + \left(s\sqrt{2}\right)^2}}{2}\right)^2}$$

$$H_S = \sqrt{3s^2 - \frac{4s^2}{4}}$$

$$H_S = \sqrt{2s^2}$$

$$\boldsymbol{H_S} = s\sqrt{2}$$

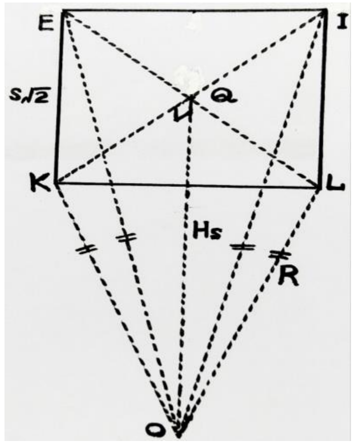

Figure 4: Square face EKLI is at a normal distance H_S from centre O of polyhedron. $\boldsymbol{OE = OK = OL = OI = R}$

Consider an equilateral triangular face EFJ of side s which is at a normal distance $OT = H_T$ from the centre O of truncated rhombic dodecahedron. Join its vertices E, F, J & centre M to the centre O of truncated rhombic dodecahedron (as shown in fig-5)

In right ΔOTF (see fig-5), using Pythagorean theorem, we get

$$OT = \sqrt{(OF)^2 - (TF)^2}$$

$$H_T = \sqrt{(R)^2 - \left(\frac{s}{\sqrt{3}}\right)^2} \qquad \left(\text{Circum radius of equilateral } \Delta = \frac{\text{side}}{\sqrt{3}}\right)$$

$$H_T = \sqrt{\left(s\sqrt{3}\right)^2 - \frac{s^2}{3}}$$

$$\boldsymbol{H_T} = \sqrt{3s^2 - \frac{s^2}{3}} = \sqrt{\frac{8s^2}{3}} = 2s\sqrt{\frac{2}{3}}$$

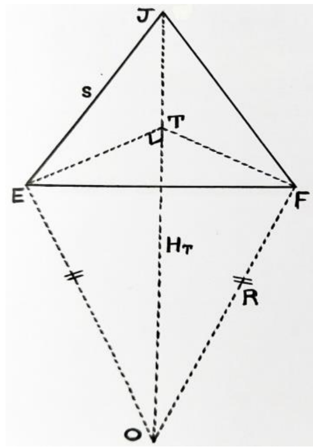

Figure 5: Equilateral triangular face EFJ is at a normal distance $\boldsymbol{H_T}$ from the centre O and $\boldsymbol{OE = OF = OJ = R}$

Hence, the **normal distances $\boldsymbol{H_R}$, $\boldsymbol{H_S}$ & $\boldsymbol{H_T}$ of rectangular, square & equilateral triangular faces respectively from the centre of truncated rhombic dodecahedron with unequal edges $\boldsymbol{s}$ & $\boldsymbol{s\sqrt{2}}$**, are given as follows

$$\boldsymbol{H_R = \frac{3s}{2}}\ , \quad \boldsymbol{H_S = s\sqrt{2} \approx 1.414213562\ s}\ \ \&\ \ \boldsymbol{H_T = 2s\sqrt{\frac{2}{3}} \approx 1.632993162\ s} \qquad \ldots\ldots\ldots\ldots(2)$$

It is clear from above values of normal distances that the equilateral triangular faces are the farthest from the centre while square faces are the closest to the centre & rectangular faces are at a normal distance between these two. For finite value of small edge length $s \Rightarrow \boldsymbol{H_S < H_R < H_T < R}$

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Surface Area (A_s) of truncated rhombic dodecahedron: The surface of a truncated rhombic dodecahedron consists of 12 congruent rectangular faces each of length $s\sqrt{2}$ & width s, 6 congruent square faces each with side $s\sqrt{2}$ and 8 congruent equilateral triangular faces each with side s . Therefore, the (total) surface area of truncated rhombic dodecahedron is given as follows

$$A_s = 12(\text{Area of rectangular face}) + 6(\text{Area of square face}) + 8(\text{Area of equi. triangular face})$$

$$= 12(s \cdot s\sqrt{2}) + 6\left(\left(s\sqrt{2}\right)^2\right) + 8\left(\frac{\sqrt{3}}{4}(s)^2\right)$$

$$= 12s^2\sqrt{2} + 12s^2 + 2s^2\sqrt{3}$$

$$= 2s^2\left(6\sqrt{2} + 6 + \sqrt{3}\right)$$

$$\therefore \textbf{Surface area},\ \boldsymbol{A_s = 2s^2\left(6\sqrt{2} + 6 + \sqrt{3}\right) \approx 32.43466436\, s^2}$$

Volume (V) of truncated rhombic dodecahedron: The surface of a truncated rhombic dodecahedron consists of 12 congruent rectangular faces each of length $s\sqrt{2}$ & width s, 6 congruent square faces each with side $s\sqrt{2}$ and 8 congruent equilateral triangular faces each with side s. Thus a solid truncated rhombic dodecahedron can assumed to consisting of 12 congruent right pyramids with rectangular base of length $s\sqrt{2}$ & width s & normal height H_R , 6 congruent right pyramids with square base of side $s\sqrt{2}$ & normal height H_S and 8 congruent right pyramids with equilateral triangular base with side $s\sqrt{2}$ & normal height H_T (as shown in above fig-1). Therefore, the volume of truncated rhombic dodecahedron is given as

$$V = 12(\text{Volume of rectangular right pyramid}) + 6(\text{Volume of square right pyramid}) + 8(\text{Volume of equilateral triangular right pyramid})$$

$$= 12\left(\frac{1}{3}(s \cdot s\sqrt{2}) \cdot \frac{3s}{2}\right) + 6\left(\frac{1}{3}\left(s\sqrt{2}\right)^2 \cdot s\sqrt{2}\right) + 8\left(\frac{1}{3}\left(\frac{\sqrt{3}}{4}s^2\right) \cdot 2s\sqrt{\frac{2}{3}}\right)$$

$$= 12\left(\frac{s^3}{\sqrt{2}}\right) + 6\left(\frac{2s^3\sqrt{2}}{3}\right) + 8\left(\frac{s^3}{3\sqrt{2}}\right)$$

$$= 6s^3\sqrt{2} + 4s^3\sqrt{2} + \frac{4s^3\sqrt{2}}{3}$$

$$= \frac{34s^3\sqrt{2}}{3}$$

$$\therefore \textbf{Volume},\ \boldsymbol{V = \frac{34s^3\sqrt{2}}{3} \approx 16.02775371 s^3}$$

Mean radius (R_m) of truncated rhombic dodecahedron: It is the radius of the sphere having a volume equal to that of a given truncated rhombic dodecahedron with edge lengths s & $s\sqrt{2}$. It is computed as follows

$$\text{volume of sphere with mean radius } R_m = \text{volume of truncated rhombic dodecahedron}$$

$$\frac{4}{3}\pi(R_m)^3 = \frac{34s^3\sqrt{2}}{3}$$

$$(R_m)^3 = \frac{17s^3}{\pi\sqrt{2}}$$

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$$R_m = \left(\frac{17s^3}{\pi\sqrt{2}}\right)^{1/3}$$

$$R_m = s\left(\frac{17}{\pi\sqrt{2}}\right)^{1/3}$$

$$\therefore \textbf{Mean radius,} \ \ R_m = s\left(\frac{17}{\pi\sqrt{2}}\right)^{1/3} \approx 1.564088599\, s \qquad \ldots\ldots\ldots\ldots(3)$$

Dihedral angle between any two faces meeting at a vertex of truncated rhombic dodecahedron:

A truncated rhombic dodecahedron has 24 identical vertices at each of which two rectangular, one square & one equilateral triangular faces meet together. We will consider each pair of two faces meeting at a vertex & find the **dihedral angle measured internally** between these two faces.

Consider a rectangular and a square faces meeting each other at the vertex E which are inclined at a dihedral angle $\boldsymbol{\theta_{RS}}$ (as shown in the fig-6). The line EF shows the small side of rectangular face EFGI (see fig-3 above) & line EK shows the side of square face EKLI (see fig-4 above). Drop the perpendiculars OM & OQ from the centre O to the rectangular & square faces at the centres M & Q which are shown by OM_1 & OQ_1 respectively in the projected view in fig-6.

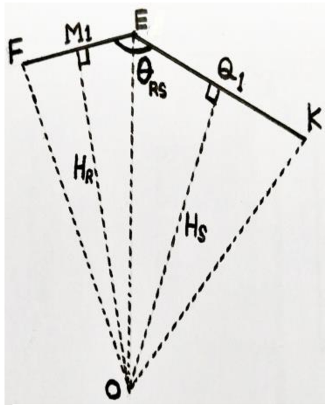

Figure 6: Dihedral angle $\angle FEK = \theta_{RS}$ between a rectangular & a square faces shown by the sides EF & EK ⊥ to the plane of paper

In right ΔOM_1E (see fig-6),

$$\tan\angle M_1EO = \frac{OM_1}{EM_1} = \frac{H_R}{\left(\frac{EF}{2}\right)} = \frac{\left(\frac{3s}{2}\right)}{\left(\frac{s}{2}\right)} = 3 \qquad (\text{since, } \ \mathrm{EF} = s)$$

$$\angle M_1EO = \tan^{-1} 3$$

In right ΔOQ_1E (see fig-6),

$$\tan\angle Q_1EO = \frac{OQ_1}{EQ_1} = \frac{H_S}{\left(\frac{EK}{2}\right)} = \frac{(s\sqrt{2})}{\left(\frac{s\sqrt{2}}{2}\right)} = 2 \qquad \left(\text{since, } \ \mathrm{EK} = s\sqrt{2}\right)$$

$$\angle Q_1EO = \tan^{-1} 2$$

$$\Rightarrow \angle FEK = \angle M_1EO + \angle Q_1EO$$

$$\theta_{RS} = \tan^{-1} 3 + \tan^{-1} 2$$

$$= \pi + \tan^{-1}\left(\frac{3+2}{1-3\cdot 2}\right) \qquad \left(\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)\ \forall\, xy > 1\right)$$

$$= \pi + \tan^{-1}(-1)$$

$$= \pi + \left(-\frac{\pi}{4}\right)$$

$$= \frac{3\pi}{4}$$

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Hence, the **dihedral angle θ_{RS} between any two adjacent rectangular & square faces of a truncated rhombic dodecahedron**, is given as follows

$$\boldsymbol{\theta_{RS} = \frac{3\pi}{4} = 135^o} \quad \ldots\ldots\ldots\ldots(4)$$

Consider a rectangular and an equilateral triangular faces meeting each other at the vertex E which are inclined at a dihedral angle $\boldsymbol{\theta_{RT}}$. The line EI shows the large side of rectangular face EFGI (see fig-3 above) & line EJ_1 shows the altitude from vertex E to the side FJ of equilateral triangular face EFJ (see fig-5 above). Drop the perpendiculars OM & OT from the centre O to these faces which are shown by OM_1 & OT_1 respectively in projected view (as shown in fig-7).

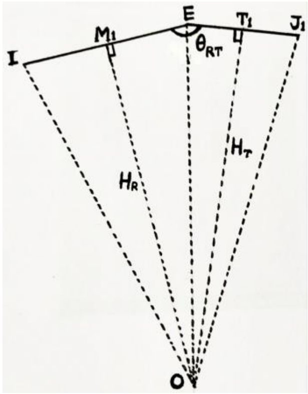

Figure 7: Dihedral angle $\angle IEJ_1 = \theta_{RT}$ between a rectangular & an equilateral triangular faces shown by the side EI & altitude $EJ_1 \perp$ to the plane of paper

In right ΔOM_1E (see fig-7),

$$\tan\angle M_1EO = \frac{OM_1}{EM_1} = \frac{H_R}{\left(\frac{EI}{2}\right)} = \frac{\left(\frac{3s}{2}\right)}{\left(\frac{s\sqrt{2}}{2}\right)} = \frac{3}{\sqrt{2}} \qquad \left(\text{since, } \ \mathrm{EI} = s\sqrt{2}\right)$$

$$\angle M_1EO = \tan^{-1}\frac{3}{\sqrt{2}}$$

In right ΔOT_1E (see fig-7),

$$\tan\angle T_1EO = \frac{OT_1}{ET_1} = \frac{H_T}{\left(\frac{EJ_1}{3}\right)} = \frac{\left(2s\sqrt{\frac{2}{3}}\right)}{\left(\frac{\frac{s\sqrt{3}}{2}}{3}\right)} = 4\sqrt{2} \qquad \left(\text{since, } \ \mathrm{E}J_1 = s\sin 60^o = \frac{s\sqrt{3}}{2}\right)$$

$$\angle T_1EO = \tan^{-1}4\sqrt{2}$$

$$\Rightarrow \angle IEJ_1 = \angle M_1EO + \angle T_1EO$$

$$\theta_{RT} = \tan^{-1}\frac{3}{\sqrt{2}} + \tan^{-1}4\sqrt{2}$$

$$= \pi + \tan^{-1}\left(\frac{\frac{3}{\sqrt{2}} + 4\sqrt{2}}{1 - \frac{3}{\sqrt{2}}\cdot 4\sqrt{2}}\right) \qquad \left(\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \ \forall\, xy > 1\right)$$

$$= \pi + \tan^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$

$$= \pi - \tan^{-1}\frac{1}{\sqrt{2}}$$

Hence, the **dihedral angle θ_{RT} between any two adjacent rectangular & equilateral triangular faces of a truncated rhombic dodecahedron**, is given as follows

$$\boldsymbol{\theta_{RT} = \pi - \tan^{-1}\frac{1}{\sqrt{2}} \approx 144^o 44' 8.2''} \quad \ldots\ldots\ldots\ldots(5)$$

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Consider a square and an equilateral triangular faces meeting each other at the vertex E which are inclined at a dihedral angle $\boldsymbol{\theta_{ST}}$. The line EL shows the diagonal of square face EKLI (see fig-4 above) & line EF_1 shows the altitude from vertex E to the side FJ of equilateral triangular face EFJ (see fig-5 above). Drop the perpendiculars OQ & OT from the centre O to the centres Q & T of square & triangular faces respectively (as shown in the projected view in fig-8).

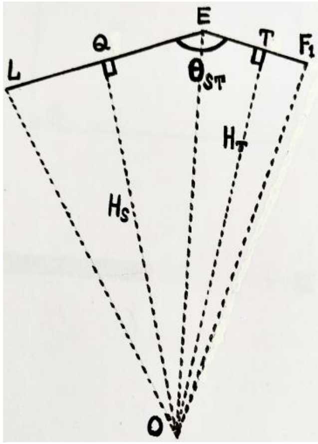

Figure 8: Dihedral angle $\angle LEF_1 = \theta_{ST}$ between a square & an equilateral triangular faces shown by the diagonal EL & altitude $EF_1 \perp$ to the plane of paper

In right ΔOQE (see fig-8),

$$\tan\angle QEO = \frac{OQ}{QE} = \frac{H_S}{\left(\frac{EL}{2}\right)} = \frac{s\sqrt{2}}{\left(\frac{2s}{2}\right)} = \sqrt{2} \qquad \left(\text{since, } \ EL = \sqrt{2}(s\sqrt{2}) = 2s\right)$$

$$\angle QEO = \tan^{-1}\sqrt{2}$$

In right ΔOTE (see fig-8),

$$\tan\angle TEO = \frac{OT}{TE} = \frac{H_T}{\left(\frac{2}{3}EF_1\right)} = \frac{\left(2s\sqrt{\frac{2}{3}}\right)}{\left(\frac{2}{3}\frac{s\sqrt{3}}{2}\right)} = 2\sqrt{2} \qquad \left(\text{since, } \ EF_1 = s\sin 60^o = \frac{s\sqrt{3}}{2}\right)$$

$$\angle TEO = \tan^{-1} 2\sqrt{2}$$

$$\Rightarrow \angle LEF_1 = \angle QEO + \angle TEO$$

$$\theta_{ST} = \tan^{-1}\sqrt{2} + \tan^{-1} 2\sqrt{2}$$

$$= \pi + \tan^{-1}\left(\frac{\sqrt{2} + 2\sqrt{2}}{1 - \sqrt{2}\cdot 2\sqrt{2}}\right) \qquad \left(\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \ \forall\, xy > 1\right)$$

$$= \pi + \tan^{-1}(-\sqrt{2})$$

$$= \pi - \tan^{-1}\sqrt{2}$$

Hence, the **dihedral angle $\boldsymbol{\theta_{ST}}$ between square & equilateral triangular faces meeting at the same vertex of truncated rhombic dodecahedron**, is given as follows

$$\boldsymbol{\theta_{ST} = \pi - \tan^{-1}\sqrt{2} \approx 125^o 15' 51.8''} \qquad \ldots\ldots\ldots\ldots(\mathbf{6})$$

Consider two congruent rectangular faces meeting each other at the vertex E which are inclined at a dihedral angle $\boldsymbol{\theta_{RR}}$. The line EG shows the diagonal of rectangular face EFGI (see fig-3 above) & line EG' shows the diagonal of another rectangular face EF'G'I'. Drop the perpendiculars OM & OM' from the centre O to the centres M & M' of these rectangular faces with no side common (as shown in the projected in fig-9).

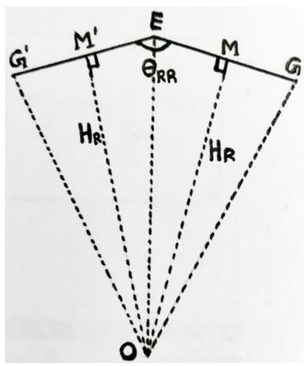

Figure 9: Dihedral angle $\angle G'EG = \theta_{RR}$ between two rectangular faces, meeting at a vertex, shown by diagonals EG & EG'⊥ to the plane of paper

In right ΔOME (see fig-9),

$$\tan\angle MEO = \frac{OM}{EM} = \frac{H_R}{\left(\frac{EG}{2}\right)} = \frac{\left(\frac{3s}{2}\right)}{\left(\frac{s\sqrt{3}}{2}\right)} = \sqrt{3} \qquad \left(EG = \sqrt{s^2 + \left(s\sqrt{2}\right)^2} = s\sqrt{3}\right)$$

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$$\angle MEO = \tan^{-1}\sqrt{3} \qquad \Rightarrow \angle M'EO = \angle MEO = \tan^{-1}\sqrt{3} = \pi/3$$

$$\Rightarrow \angle G'EG = \angle M'EO + \angle MEO$$

$$\theta_{RR} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

Hence, the **dihedral angle $\boldsymbol{\theta_{RR}}$ between any two rectangular faces meeting at the same vertex of truncated rhombic dodecahedron**, is given as follows

$$\boldsymbol{\theta_{RR} = \frac{2\pi}{3} = 120^o} \qquad \dots\dots\dots\dots(7)$$

Solid angles $\boldsymbol{\omega_R, \omega_S}$ & $\boldsymbol{\omega_T}$ subtended by rectangular, square & equilateral triangular faces respectively at the centre of truncated rhombic dodecahedron:

We know that the solid angle (ω), subtended by a rectangular plane of length l & width b at any point lying at a distance h on the perpendicular axis passing through the centre, is given by "**HCR's Theory of Polygon**" as follows

$$\boldsymbol{\omega = 4\sin^{-1}\left(\frac{lb}{\sqrt{(l^2+4h^2)(b^2+4h^2)}}\right)}$$

Substituting the corresponding values in above formula i.e. length $= s\sqrt{2}$, width $b = s$ & normal height $h = H_R = 3s/2$, the solid angle ω_R subtended by the rectangular face EFGI at the centre O of truncated rhombic dodecahedron (as shown in above fig-3), is obtained as follows

$$\omega_R = 4\sin^{-1}\left(\frac{s\sqrt{2}\cdot s}{\sqrt{\left((s\sqrt{2})^2 + 4\left(\frac{3s}{2}\right)^2\right)\left(s^2 + 4\left(\frac{3s}{2}\right)^2\right)}}\right)$$

$$\boldsymbol{\omega_R} = 4\sin^{-1}\left(\frac{s^2\sqrt{2}}{\sqrt{(11s^2)(10s^2)}}\right) = 4\sin^{-1}\left(\frac{s^2\sqrt{2}}{s^2\sqrt{110}}\right) = 4\sin^{-1}\left(\sqrt{\frac{2}{110}}\right) = 4\sin^{-1}\left(\sqrt{\frac{1}{55}}\right)$$

Hence, the **solid angle $\boldsymbol{\omega_R}$ subtended by any of 12 congruent rectangular faces at the centre of a truncated rhombic dodecahedron**, is given as follows

$$\boldsymbol{\omega_R = 4\sin^{-1}\left(\frac{1}{\sqrt{55}}\right)\ \mathrm{sr} \approx 0.541007833\ \mathrm{sr}} \qquad \dots\dots\dots\dots(8)$$

Similarly, substituting the corresponding values in above formula i.e. length $= s\sqrt{2}$, width $b = s\sqrt{2}$ & normal height $h = H_S = s\sqrt{2}$, the solid angle ω_S subtended by the square face EKLI at the centre O of truncated rhombic dodecahedron (as shown in above fig-4), is obtained as follows

$$\omega_S = 4\sin^{-1}\left(\frac{s\sqrt{2}\cdot s\sqrt{2}}{\sqrt{\left((s\sqrt{2})^2 + 4(s\sqrt{2})^2\right)\left(s^2 + 4(s\sqrt{2})^2\right)}}\right)$$

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$$\omega_S = 4\sin^{-1}\left(\frac{2s^2}{\sqrt{(10s^2)(10s^2)}}\right) = 4\sin^{-1}\left(\frac{2s^2}{10s^2}\right) = 4\sin^{-1}(0.2)$$

Hence, the **solid angle ω_S subtended by any of 6 congruent square faces at the centre of a truncated rhombic dodecahedron**, is given as follows

$$\boldsymbol{\omega_S = 4\sin^{-1}(0.2)\ \text{sr} \approx 0.805431683\ \text{sr}} \quad \ldots\ldots\ldots\ldots(9)$$

We know that the solid angle (ω) subtended by any regular polygonal plane with n no. of sides each of length a at any point lying at a distance H on the vertical axis passing through the centre, is given by "**HCR's Theory of Polygon**" as follows

$$\omega = 2\pi - 2n\sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2 + a^2\cot^2\frac{\pi}{n}}}\right)$$

Substituting the corresponding values in above formula i.e. number of sides $n = 3$ (for regular Δ), length of each side $a = s$, & normal height $H = H_T = 2s\sqrt{2/3}$, the solid angle ω_T subtended by the equilateral triangular face EFJ at the centre O of truncated rhombic dodecahedron (as shown in above fig-5), is obtained as follows

$$\omega_T = 2\pi - 2\times 3\sin^{-1}\left(\frac{2\left(2s\sqrt{\frac{2}{3}}\right)\sin\frac{\pi}{3}}{\sqrt{4\left(2s\sqrt{\frac{2}{3}}\right)^2 + s^2\cot^2\frac{\pi}{3}}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{4s\sqrt{\frac{2}{3}}\times\frac{\sqrt{3}}{2}}{\sqrt{\frac{32s^2}{3} + \frac{s^2}{3}}}\right) = 2\pi - 6\sin^{-1}\left(\frac{2s\sqrt{2}}{\sqrt{\frac{33s^2}{3}}}\right) = 2\pi - 6\sin^{-1}\left(\frac{2\sqrt{2}}{\sqrt{11}}\right)$$

Hence, the **solid angle ω_T subtended by any of 8 congruent equilateral triangular faces at the centre of a truncated rhombic dodecahedron**, is given as follows

$$\boldsymbol{\omega_T = 2\pi - 6\sin^{-1}\left(2\sqrt{\frac{2}{11}}\right)\ \text{sr} \approx 0.155210814\ \text{sr}} \quad \ldots\ldots\ldots\ldots(10)$$

Total solid angle: We know that a truncated rhombic dodecahedron has 12 congruent rectangular faces, 6 congruent square faces & 8 congruent equilateral triangular faces. Therefore, the total solid angle subtended by all the faces at the centre of a truncated rhombic dodecahedron, is given as follows

$$\omega = 12(\omega_R) + 6(\omega_S) + 8(\omega_T)$$

$$\omega = 12\left(4\sin^{-1}\left(\frac{1}{\sqrt{55}}\right)\right) + 6(4\sin^{-1}(0.2)) + 8\left(2\pi - 6\sin^{-1}\left(2\sqrt{\frac{2}{11}}\right)\right)$$

$$\omega = 48\sin^{-1}\left(\frac{1}{\sqrt{55}}\right) + 24\sin^{-1}(0.2) + 16\pi - 48\sin^{-1}\left(2\sqrt{\frac{2}{11}}\right)$$

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$$\omega = 16\pi + 24\sin^{-1}(0.2) - 48\left(\sin^{-1}\left(2\sqrt{\frac{2}{11}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{55}}\right)\right)$$

$$\omega = 16\pi + 24\sin^{-1}\left(\frac{1}{5}\right) - 48\left(\sin^{-1}\left(\sqrt{\frac{8}{11}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{55}}\right)\right)$$

Using formula: $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) \quad \forall\ |x|,|y| \in [0,1],$

$$\omega = 16\pi + 48\left(\frac{1}{2}\sin^{-1}\left(\frac{1}{5}\right)\right) - 48\left(\sin^{-1}\left(\sqrt{\frac{8}{11}}\sqrt{\frac{54}{55}} - \sqrt{\frac{3}{11}}\frac{1}{\sqrt{55}}\right)\right)$$

Using formula: $\frac{1}{2}\sin^{-1}x = \sin^{-1}\left(\sqrt{\frac{1-\sqrt{1-x^2}}{2}}\right) \quad \forall\ |x| \in [0,1],$

$$\omega = 16\pi + 48\left(\sin^{-1}\left(\sqrt{\frac{1-\sqrt{1-\left(\frac{1}{5}\right)^2}}{2}}\right)\right) - 48\left(\sin^{-1}\left(\frac{12\sqrt{3}}{11\sqrt{5}} - \frac{\sqrt{3}}{11\sqrt{5}}\right)\right)$$

$$\omega = 16\pi + 48\sin^{-1}\left(\sqrt{\frac{1-\frac{2\sqrt{6}}{5}}{2}}\right) - 48\left(\sin^{-1}\left(\frac{12\sqrt{3}-\sqrt{3}}{11\sqrt{5}}\right)\right)$$

$$\omega = 16\pi + 48\sin^{-1}\left(\sqrt{\frac{5-2\sqrt{6}}{10}}\right) - 48\left(\sin^{-1}\left(\frac{11\sqrt{3}}{11\sqrt{5}}\right)\right)$$

$$\omega = 16\pi + 48\sin^{-1}\left(\sqrt{\frac{\left(\sqrt{3}-\sqrt{2}\right)^2}{10}}\right) - 48\left(\sin^{-1}\left(\frac{\sqrt{3}}{\sqrt{5}}\right)\right)$$

$$\omega = 16\pi + 48\sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{10}}\right) - 48\sin^{-1}\left(\sqrt{\frac{3}{5}}\right)$$

$$\omega = 16\pi - 48\left(\sin^{-1}\left(\sqrt{\frac{3}{5}}\right) - \sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{10}}\right)\right)$$

$$\omega = 16\pi - 48\left(\sin^{-1}\left(\sqrt{\frac{3}{5}}\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{10}}\right) - \sqrt{\frac{2}{5}}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{10}}\right)\right)\right)$$

$$\omega = 16\pi - 48\left(\sin^{-1}\left(\frac{\sqrt{3}\left(\sqrt{3}+\sqrt{2}\right)}{5\sqrt{2}} - \frac{\sqrt{2}\left(\sqrt{3}-\sqrt{2}\right)}{5\sqrt{2}}\right)\right)$$

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$$\omega = 16\pi - 48\sin^{-1}\left(\frac{3+\sqrt{6}}{5\sqrt{2}} - \frac{\sqrt{6}-2}{5\sqrt{2}}\right)$$

$$\omega = 16\pi - 48\sin^{-1}\left(\frac{3+\sqrt{6}-\sqrt{6}+2}{5\sqrt{2}}\right)$$

$$\omega = 16\pi - 48\sin^{-1}\left(\frac{5}{5\sqrt{2}}\right)$$

$$\omega = 16\pi - 48\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\omega = 16\pi - 48\left(\frac{\pi}{4}\right)$$

$$\omega = 16\pi - 12\pi$$

$$\boldsymbol{\omega = 4\pi}\ sr$$

Above result shows that the solid angle subtended by a truncated rhombic dodecahedron at its centre is $\boldsymbol{4\pi}\ sr$ It is true that the **solid angle, subtended by any closed surface at any point inside it, is always $\boldsymbol{4\pi\ sr}$**

We know that a truncated rhombic dodecahedron has total 24 identical vertices at each of which two rectangular, one square & one equilateral triangular faces meet together. Thus we would analyse one of 24 identical vertices to compute solid angle subtended by a truncated rhombic dodecahedron at the same vertex.

Solid angle subtended by truncated rhombic dodecahedron at any of its 24 identical vertices:

Consider any of 24 identical vertices say vertex P of truncated rhombic dodecahedron. Join the end points A, B, C & D of the edges AP, BP, CP & DP, meeting at vertex P, to get a (plane) trapezium ABCD of sides $AB = s$, $BC = AD = s\sqrt{3}$ & $CD = 2s$ (as shown in fig-10).

Join the foot point Q of perpendicular PQ drawn from vertex P to the plane of trapezium ABCD, to the vertices A, B, C & D . Drop the perpendiculars MN & QE from midpoint M & foot point Q to the sides CD & BC respectively of trapezium ABCD (as shown in the fig-11)

We have found out that dihedral angle between square & equilateral triangular faces is $\angle MPN = \pi - \tan^{-1}\sqrt{2}$. The perpendicular PQ dropped from the vertex P to the plane of trapezium ABCD will fall at the point Q lying on the line MN (as shown in the fig-11).

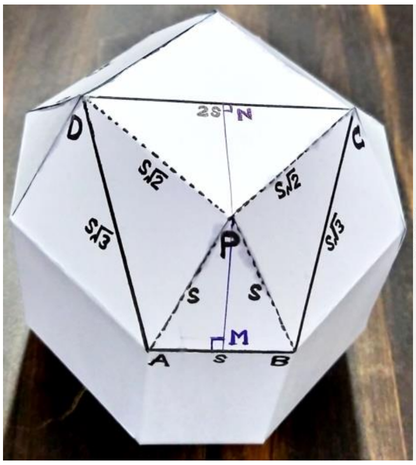

Figure 10: A trapezium ABCD is formed by joining the endpoints A, B, C & D of edges AP, BP, CP & DP meeting at the vertex P of given polyhedron

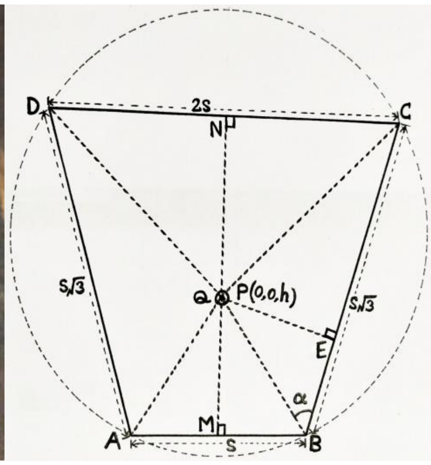

Figure 11: Point Q is the foot of perpendicular PQ drawn from the vertex P to the plane of trapezium ABCD. Point P is lying at a normal height h from the point foot Q (⊥ to the plane of paper).

In ΔMPN (see fig-12 below), using cosine formula as follows

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$$\cos\angle MPN = \frac{(PM)^2 + (PN)^2 - (MN)^2}{2(PM)(PN)}$$

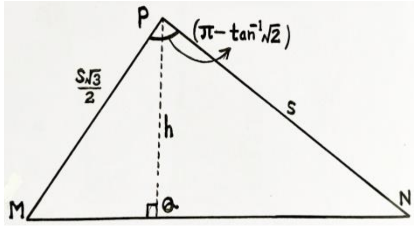

Figure 12: Perpendicular PQ dropped from vertex P to the plane of trapezium ABCD falls at the point Q on the line MN

$$\cos(\pi - \tan^{-1}\sqrt{2}) = \frac{\left(\frac{s\sqrt{3}}{2}\right)^2 + (s)^2 - (MN)^2}{2\left(\frac{s\sqrt{3}}{2}\right)(s)} \Rightarrow -\cos(\tan^{-1}\sqrt{2}) = \frac{\frac{3s^2}{4} + s^2 - MN^2}{s^2\sqrt{3}}$$

$$-\cos\left(\cos^{-1}\frac{1}{\sqrt{3}}\right) = \frac{\frac{7s^2}{4} - MN^2}{s^2\sqrt{3}} \quad \Rightarrow -\frac{1}{\sqrt{3}} = \frac{\frac{7s^2}{4} - MN^2}{s^2\sqrt{3}} \quad \Rightarrow -s^2 = \frac{7s^2}{4} - MN^2$$

$$\Rightarrow \ MN^2 = \frac{7s^2}{4} + s^2 \ \Rightarrow \ MN^2 = \frac{11s^2}{4} \quad \Rightarrow \ \boldsymbol{MN = \frac{s\sqrt{11}}{2}}$$

Now, the area of ΔMPN (see fig-12), is given as follows

$$\frac{1}{2}(MN)(PQ) = \frac{1}{2}(PM)(PN)\sin(\pi - \tan^{-1}\sqrt{2})$$

$$\left(\frac{s\sqrt{11}}{2}\right)(PQ) = \left(\frac{s\sqrt{3}}{2}\right)(s)\sin(\tan^{-1}\sqrt{2})$$

$$\sqrt{11}PQ = s\sqrt{3}\sin\left(\sin^{-1}\sqrt{\frac{2}{3}}\right) \quad \Rightarrow \quad \boldsymbol{PQ} = \frac{s\sqrt{3}}{\sqrt{11}}\sqrt{\frac{2}{3}} = \boldsymbol{s\sqrt{\frac{2}{11}}}$$

Using Pythagorean theorem in right ΔPQM (see above fig-12 above) , we get

$$\boldsymbol{MQ} = \sqrt{(PM)^2 - (PQ)^2} = \sqrt{\left(\frac{s\sqrt{3}}{2}\right)^2 - \left(s\sqrt{\frac{2}{11}}\right)^2} = \sqrt{\frac{3s^2}{4} - \frac{2s^2}{11}} = \sqrt{\frac{25s^2}{44}} = \boldsymbol{\frac{5s}{2\sqrt{11}}}$$

$$\Rightarrow \ \boldsymbol{QN} = MN - MQ = \frac{s\sqrt{11}}{2} - \frac{5s}{2\sqrt{11}} = \boldsymbol{\frac{3s}{\sqrt{11}}}$$

Using Pythagorean theorem in right ΔQMB (see above fig-11) , we get

$$\boldsymbol{BQ} = \sqrt{(MQ)^2 + (MB)^2} = \sqrt{\left(\frac{5s}{2\sqrt{11}}\right)^2 + \left(\frac{s}{2}\right)^2} = \sqrt{\frac{25s^2}{44} + \frac{s^2}{4}} = \sqrt{\frac{36s^2}{44}} = \boldsymbol{\frac{3s}{\sqrt{11}}}$$

Using Pythagorean theorem in right ΔPQC (see above fig-13) , we get

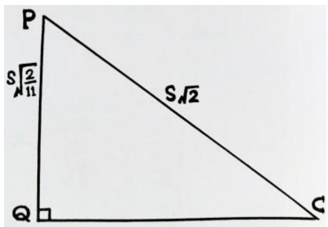

Figure 13: Right ΔPQC is obtained by dropping $\perp$ to the plane of trapezium ABCD

$$\boldsymbol{QC} = \sqrt{(PC)^2 - (PQ)^2} = \sqrt{(s\sqrt{2})^2 - \left(s\sqrt{\frac{2}{11}}\right)^2} = \sqrt{2s^2 - \frac{2s^2}{11}} = \sqrt{\frac{20s^2}{11}} = \boldsymbol{2s\sqrt{\frac{5}{11}}}$$

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In ΔQBC (see fig-11 above), using cosine formula as follows

$$\cos\alpha=\frac{(BQ)^2+(BC)^2-(QC)^2}{2(BQ)(BC)}=\frac{\left(\frac{3s}{\sqrt{11}}\right)^2+\left(s\sqrt{3}\right)^2-\left(2s\sqrt{\frac{5}{11}}\right)^2}{2\left(\frac{3s}{\sqrt{11}}\right)\left(s\sqrt{3}\right)}=\frac{\frac{9s^2}{11}+3s^2-\frac{20s^2}{11}}{\frac{6s^2\sqrt{3}}{\sqrt{11}}}$$

$$\boldsymbol{\cos\alpha}=\frac{\frac{9s^2+33s^2-20s^2}{11}}{\frac{6s^2\sqrt{3}}{\sqrt{11}}}=\frac{\frac{22s^2}{11}}{\frac{6s^2\sqrt{3}}{\sqrt{11}}}=\frac{\sqrt{11}}{3\sqrt{3}}=\boldsymbol{\frac{1}{3}\sqrt{\frac{11}{3}}}$$

In right ΔBEQ (see above fig-11),

$$\cos\alpha=\frac{BE}{BQ}\quad\Rightarrow\ \boldsymbol{BE}=BQ\ \cos\alpha=\frac{3s}{\sqrt{11}}\cdot\frac{1}{3}\sqrt{\frac{11}{3}}=\boldsymbol{\frac{s}{\sqrt{3}}}$$

$$\Rightarrow\ \boldsymbol{EC}=BC-BE=s\sqrt{3}-\frac{s}{\sqrt{3}}=\boldsymbol{\frac{2s}{\sqrt{3}}}$$

Using Pythagorean theorem in right ΔBEQ (see above fig-11) , we get

$$\boldsymbol{QE}=\sqrt{(BQ)^2-(BE)^2}=\sqrt{\left(\frac{3s}{\sqrt{11}}\right)^2-\left(\frac{s}{\sqrt{3}}\right)^2}=\sqrt{\frac{9s^2}{11}-\frac{s^2}{3}}=\sqrt{\frac{16s^2}{33}}=\boldsymbol{\frac{4s}{\sqrt{33}}}$$

We know from **HCR's Theory of Polygon** that the **solid angle (ω), subtended by a right triangle OGH having perpendicular p & base b at any point P at a normal distance h on the vertical axis passing through the vertex O** (as shown in the fig-14)**,** is given by **HCR's Standard Formula-1** as follows

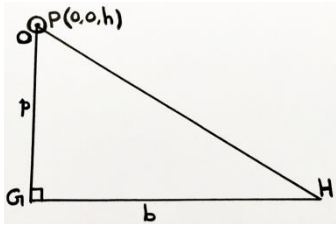

Figure-14: Point P lies at normal height h from vertex O of right ΔOGH (⊥ to plane of paper).

$$\omega=\sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right)-\sin^{-1}\left(\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right)$$

Now, the solid angle $\omega_{\Delta QMB}$ subtended by right ΔQMB at the vertex P (see above fig-11) is obtained by substituting the corresponding values (as derived above) in above standard-1 formula i.e. base $b=MB=\frac{s}{2}$, perpendicular $p=MQ=\frac{5s}{2\sqrt{11}}$ & normal height $h=PQ=s\sqrt{\frac{2}{11}}$ as follows

$$\omega_{\Delta QMB}=\sin^{-1}\left(\frac{\frac{s}{2}}{\sqrt{\left(\frac{s}{2}\right)^2+\left(\frac{5s}{2\sqrt{11}}\right)^2}}\right)-\sin^{-1}\left(\left(\frac{\frac{s}{2}}{\sqrt{\left(\frac{s}{2}\right)^2+\left(\frac{5s}{2\sqrt{11}}\right)^2}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{\sqrt{\left(s\sqrt{\frac{2}{11}}\right)^2+\left(\frac{5s}{2\sqrt{11}}\right)^2}}\right)\right)$$

$$\omega_{\Delta QMB}=\sin^{-1}\left(\frac{\frac{s}{2}}{\frac{3s}{\sqrt{11}}}\right)-\sin^{-1}\left(\left(\frac{\frac{s}{2}}{\frac{3s}{\sqrt{11}}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{\frac{s\sqrt{3}}{2}}\right)\right)=\sin^{-1}\left(\frac{\sqrt{11}}{6}\right)-\sin^{-1}\left(\left(\frac{\sqrt{11}}{6}\right)\left(2\sqrt{\frac{2}{33}}\right)\right)$$

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$$\omega_{\Delta QMB} = \sin^{-1}\left(\frac{\sqrt{11}}{6}\right) - \sin^{-1}\left(\frac{1}{3}\sqrt{\frac{2}{3}}\right)$$

Similarly, the solid angle $\omega_{\Delta BEQ}$ subtended by right ΔBEQ at the vertex P (see above fig-11) is obtained by substituting the corresponding values (as derived above) in above standard formula-1 i.e. base $b = BE = \frac{s}{\sqrt{3}}$, perpendicular $p = QE = \frac{4s}{\sqrt{33}}$ & normal height $h = PQ = s\sqrt{\frac{2}{11}}$ as follows

$$\omega_{\Delta BEQ} = \sin^{-1}\left(\frac{\frac{s}{\sqrt{3}}}{\sqrt{\left(\frac{s}{\sqrt{3}}\right)^2 + \left(\frac{4s}{\sqrt{33}}\right)^2}}\right) - \sin^{-1}\left(\left(\frac{\frac{s}{\sqrt{3}}}{\sqrt{\left(\frac{s}{\sqrt{3}}\right)^2 + \left(\frac{4s}{\sqrt{33}}\right)^2}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{\sqrt{\left(s\sqrt{\frac{2}{11}}\right)^2 + \left(\frac{4s}{\sqrt{33}}\right)^2}}\right)\right)$$

$$\omega_{\Delta BEQ} = \sin^{-1}\left(\frac{\frac{s}{\sqrt{3}}}{\frac{3s}{\sqrt{11}}}\right) - \sin^{-1}\left(\left(\frac{\frac{s}{\sqrt{3}}}{\frac{3s}{\sqrt{11}}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{s\sqrt{\frac{2}{3}}}\right)\right) = \sin^{-1}\left(\frac{1}{3}\sqrt{\frac{11}{3}}\right) - \sin^{-1}\left(\left(\frac{1}{3}\sqrt{\frac{11}{3}}\right)\left(\sqrt{\frac{3}{11}}\right)\right)$$

$$\omega_{\Delta BEQ} = \sin^{-1}\left(\frac{1}{3}\sqrt{\frac{11}{3}}\right) - \sin^{-1}\left(\frac{1}{3}\right)$$

Similarly, the solid angle $\omega_{\Delta CEQ}$ subtended by right ΔCEQ at the vertex P (see above fig-11) is obtained by substituting the corresponding values (as derived above) in above standard formula-1 i.e. base $b = EC = \frac{2s}{\sqrt{3}}$, perpendicular $p = QE = \frac{4s}{\sqrt{33}}$ & normal height $h = PQ = s\sqrt{\frac{2}{11}}$ as follows

$$\omega_{\Delta CEQ} = \sin^{-1}\left(\frac{\frac{2s}{\sqrt{3}}}{\sqrt{\left(\frac{2s}{\sqrt{3}}\right)^2 + \left(\frac{4s}{\sqrt{33}}\right)^2}}\right) - \sin^{-1}\left(\left(\frac{\frac{2s}{\sqrt{3}}}{\sqrt{\left(\frac{2s}{\sqrt{3}}\right)^2 + \left(\frac{4s}{\sqrt{33}}\right)^2}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{\sqrt{\left(s\sqrt{\frac{2}{11}}\right)^2 + \left(\frac{4s}{\sqrt{33}}\right)^2}}\right)\right)$$

$$\omega_{\Delta CEQ} = \sin^{-1}\left(\frac{\frac{2s}{\sqrt{3}}}{2s\sqrt{\frac{5}{11}}}\right) - \sin^{-1}\left(\left(\frac{\frac{2s}{\sqrt{3}}}{2s\sqrt{\frac{5}{11}}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{s\sqrt{\frac{2}{3}}}\right)\right) = \sin^{-1}\left(\sqrt{\frac{11}{15}}\right) - \sin^{-1}\left(\left(\sqrt{\frac{11}{15}}\right)\left(\sqrt{\frac{3}{11}}\right)\right)$$

$$\omega_{\Delta CEQ} = \sin^{-1}\left(\sqrt{\frac{11}{15}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

Similarly, the solid angle $\omega_{\Delta QNC}$ subtended by right ΔQNC at the vertex P (see above fig-11) is obtained by substituting the corresponding values (as derived above) in above standard-1 formula i.e. base $b = NC = s$, perpendicular $p = QN = \frac{3s}{\sqrt{11}}$ & normal height $h = PQ = s\sqrt{\frac{2}{11}}$ as follows

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$$\omega_{\Delta QNC}=\sin^{-1}\left(\frac{s}{\sqrt{(s)^2+\left(\frac{3s}{\sqrt{11}}\right)^2}}\right)-\sin^{-1}\left(\left(\frac{s}{\sqrt{(s)^2+\left(\frac{3s}{\sqrt{11}}\right)^2}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{\sqrt{\left(s\sqrt{\frac{2}{11}}\right)^2+\left(\frac{3s}{\sqrt{11}}\right)^2}}\right)\right)$$

$$\omega_{\Delta QNC}=\sin^{-1}\left(\frac{s}{2s\sqrt{\frac{5}{11}}}\right)-\sin^{-1}\left(\left(\frac{s}{2s\sqrt{\frac{5}{11}}}\right)\left(\frac{s\sqrt{\frac{2}{11}}}{s}\right)\right)=\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{5}}\right)-\sin^{-1}\left(\left(\frac{1}{2}\sqrt{\frac{11}{5}}\right)\left(\sqrt{\frac{2}{11}}\right)\right)$$

$$\omega_{\Delta QNC}=\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{5}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)$$

Now, according to **HCR's Theory of Polygon**, the solid angle ω_{MBCN} subtended by the trapezium MBCN at the vertex P (see above fig-11) is the algebraic sum of solid angles subtended by the right triangles $\Delta QMB, \Delta BEQ, \Delta CEQ$ & ΔQNC which is given as follows

$$\omega_{MBCN}=\omega_{\Delta QMB}+\omega_{\Delta BEQ}+\omega_{\Delta CEQ}+\omega_{\Delta QNC}$$

Substituting the corresponding values of solid angles (derived above) as follows

$$\omega_{MBCN}=\left(\sin^{-1}\left(\frac{\sqrt{11}}{6}\right)-\sin^{-1}\left(\frac{1}{3}\sqrt{\frac{2}{3}}\right)\right)+\left(\sin^{-1}\left(\frac{1}{3}\sqrt{\frac{11}{3}}\right)-\sin^{-1}\left(\frac{1}{3}\right)\right)$$
$$+\left(\sin^{-1}\left(\sqrt{\frac{11}{15}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)\right)+\left(\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{5}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)\right)$$

$$=\left(\sin^{-1}\left(\frac{\sqrt{11}}{6}\right)+\sin^{-1}\left(\frac{1}{3}\sqrt{\frac{11}{3}}\right)\right)-\left(\sin^{-1}\left(\frac{1}{3}\sqrt{\frac{2}{3}}\right)+\sin^{-1}\left(\frac{1}{3}\right)\right)+\left(\sin^{-1}\left(\sqrt{\frac{11}{15}}\right)+\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{5}}\right)\right)$$
$$-\left(\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\sin^{-1}\left(\frac{1}{\sqrt{10}}\right)\right)$$

Using formula: $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\quad\forall\ |x|,|y|\in[0,1]$,

$$=\left(\sin^{-1}\left(\frac{\sqrt{11}}{6}\cdot\frac{4}{3\sqrt{3}}+\frac{5}{6}\cdot\frac{1}{3}\sqrt{\frac{11}{3}}\right)\right)-\left(\sin^{-1}\left(\frac{1}{3}\sqrt{\frac{2}{3}}\cdot\frac{2\sqrt{2}}{3}+\frac{5}{3\sqrt{3}}\cdot\frac{1}{3}\right)\right)$$
$$+\left(\pi-\sin^{-1}\left(\sqrt{\frac{11}{15}}\cdot\frac{3}{2\sqrt{5}}+\frac{2}{\sqrt{15}}\cdot\frac{1}{2}\sqrt{\frac{11}{5}}\right)\right)-\left(\sin^{-1}\left(\frac{1}{\sqrt{5}}\cdot\frac{3}{\sqrt{10}}+\frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{10}}\right)\right)$$

$$=\sin^{-1}\left(\frac{4\sqrt{11}}{18\sqrt{3}}+\frac{5\sqrt{11}}{18\sqrt{3}}\right)-\sin^{-1}\left(\frac{4}{9\sqrt{3}}+\frac{5}{9\sqrt{3}}\right)+\left(\pi-\sin^{-1}\left(\frac{3\sqrt{11}}{10\sqrt{3}}+\frac{2\sqrt{11}}{10\sqrt{3}}\right)\right)-\sin^{-1}\left(\frac{3}{5\sqrt{2}}+\frac{2}{5\sqrt{2}}\right)$$

$$=\sin^{-1}\left(\frac{9\sqrt{11}}{18\sqrt{3}}\right)-\sin^{-1}\left(\frac{9}{9\sqrt{3}}\right)+\left(\pi-\sin^{-1}\left(\frac{5\sqrt{11}}{10\sqrt{3}}\right)\right)-\sin^{-1}\left(\frac{5}{5\sqrt{2}}\right)$$

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$$= \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{3}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) + \left(\pi - \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{3}}\right)\right) - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{3}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) + \pi - \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{11}{3}}\right) - \frac{\pi}{4}$$

$$= \frac{3\pi}{4} - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad \Rightarrow \boldsymbol{\omega_{MBCN} = \frac{3\pi}{4} - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)}$$

Thus, using symmetry in trapezium ABCD (see above fig-11), the solid angle ω_{ABCD} subtended by the trapezium ABCD at the vertex P of truncated rhombic dodecahedron will be twice the solid angle ω_{MBCN} subtended by the trapezium MBCN at the vertex P, as follows

$$\omega_{ABCD} = 2\omega_{MBCN} = 2\left(\frac{3\pi}{4} - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) = \frac{3\pi}{2} - 2\sin^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{3\pi}{2} - \sin^{-1}\left(2\cdot\frac{1}{\sqrt{3}}\cdot\sqrt{\frac{2}{3}}\right)$$

$$= \frac{3\pi}{2} - \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \pi + \left(\frac{\pi}{2} - \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right) = \pi + \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \pi + \sin^{-1}\left(\frac{1}{3}\right)$$

It's worth noticing that the solid angle ω_V subtended by truncated rhombic dodecahedron at its vertex P will be equal to the solid angle ω_{ABCD} subtended by the trapezium ABCD at the vertex P.

Hence, the **solid angles $\boldsymbol{\omega_V}$ subtended by a truncated rhombic dodecahedron at any of its 24 identical vertices** (at each of which two rectangular, one square & one regular triangular faces meet), is given as follows

$$\boldsymbol{\omega_V = \pi + \sin^{-1}\left(\frac{1}{3}\right)\ \text{sr} \approx 3.481429563\ \text{sr}} \qquad \ldots\ldots\ldots\ldots(\mathbf{11})$$

Paper model of a truncated rhombic dodecahedron: In order to make the paper model of a truncated rhombic dodecahedron having 12 congruent rectangular faces, 6 congruent square faces & 8 congruent equilateral triangular faces, it first requires the net of 26 faces to be drawn on a paper sheet

1: Prepare a net of 26 faces out of which there are 12 congruent rectangular faces each with length $s\sqrt{2}$ & width s, 6 congruent square faces each with side $s\sqrt{2}$ & 8 congruent equilateral triangular faces each with side s on the plain sheet of paper (as shown by left image in fig-15)

2: Fold each of 26 faces about its common (junction) edge such that the open edges of faces overlap one another & thus the net conforms to a closed surface. Glue the faces at the coincident edges to retain the shape of a truncated rhombic dodecahedron. (as shown by right image in fig-15)

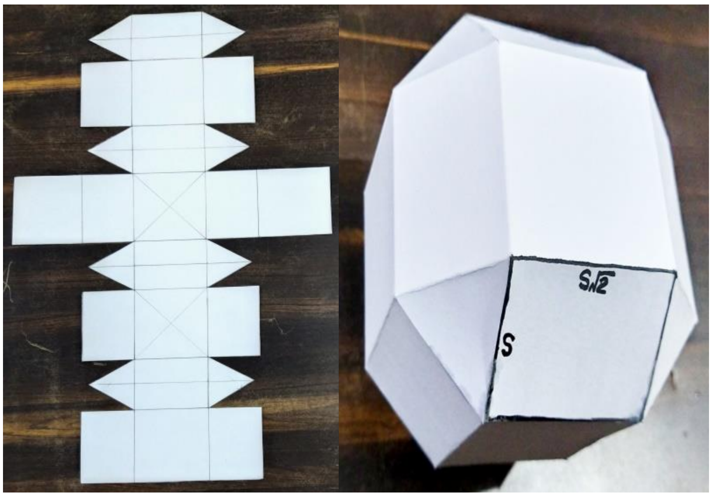

Figure 15: A net (left) of 12 congruent rectangular, 6 congruent square & 8 congruent equilateral triangular faces, is folded to conform to the shape of a truncated rhombic dodecahedron (right).

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Summary: Let there be a truncated rhombic dodecahedron having 12 congruent rectangular faces each with length $s\sqrt{2}$ & width s, 6 congruent square faces each with side $s\sqrt{2}$ & 8 congruent equilateral triangular faces each with side s, 48 edges & 24 identical vertices then all its important parameters are determined as tabulated below

Radius(R) of circumscribed sphere passing through all 24 vertices	$s\sqrt{3} \approx 1.732050808\, s$
Normal distances H_R, H_S & H_T of rectangular, square & regular triangular faces from the centre of truncated rhombic dodecahedron	$H_R = \frac{3s}{2}$, $H_S = s\sqrt{2} \approx 1.414213562$ s & $H_T = 2s\sqrt{\frac{2}{3}} \approx 1.632993162$ s
Surface area (A_s)	$A_s = 2s^2(6\sqrt{2}+6+\sqrt{3}) \approx 32.43466436\, s^2$
Volume (V)	$V = \frac{34s^3\sqrt{2}}{3} \approx 16.02775371s^3$
Mean radius (R_m) or radius of sphere having volume equal to that of the truncated rhombic dodecahedron	$R_m = s\left(\frac{17}{\pi\sqrt{2}}\right)^{1/3} \approx 1.564088599\, s$
Dihedral angle θ_{RS} between any two adjacent rectangular & square faces	$\theta_{RS} = \frac{3\pi}{4} = 135^o$
Dihedral angle θ_{RT} between any two adjacent rectangular & square faces	$\theta_{RT} = \pi - \tan^{-1}\frac{1}{\sqrt{2}} \approx 144^o 44' 8.2''$
Dihedral angle θ_{ST} between square & equilateral triangular faces meeting at the same vertex	$\theta_{ST} = \pi - \tan^{-1}\sqrt{2} \approx 125^o 15' 51.8''$
Dihedral angle θ_{RR} between any two rectangular faces meeting at the same vertex	$\theta_{RR} = \frac{2\pi}{3} = 120^o$
Solid angles ω_R , ω_S & ω_T subtended by rectangular, square & equilateral triangular faces at the centre of truncated rhombic dodecahedron	$\omega_R = 4\sin^{-1}\left(\frac{1}{\sqrt{55}}\right)$ sr ≈ 0.541 sr , $\omega_S = 4\sin^{-1}(0.2)$ sr ≈ 0.805 sr $\omega_T = 2\pi - 6\sin^{-1}\left(2\sqrt{\frac{2}{11}}\right)$ sr ≈ 0.155210814 sr
Solid angle ω_V subtended by truncated rhombic dodecahedron at its vertex	$\omega_V = \pi + \sin^{-1}\left(\frac{1}{3}\right)$ sr ≈ 3.481429563 sr

Note: Above articles had been *derived & illustrated* by **Mr H.C. Rajpoot (M Tech, Production Engineering)**

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