$Mathematical\ Analysis\ of\ (Small)\ Rhombicuboctahedron$

$(Application\ of\ HCR's\ Theory\ of\ Polygon)$

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Introduction: A rhombicuboctahedron (also called small rhombicuboctahedron) is an Archimedean solid which has 8 congruent equilateral triangular & 18 congruent square faces each having equal edge length, 48 edges & 24 vertices. It is created/generated either by shifting/translating all 8 equilateral triangular faces of a regular octahedron radially outwards by the same distance without any other transformation (i.e. rotation, distortion etc.) or by shifting/translating all 6 square faces of a cube (regular hexahedron) radially outwards by the same distance without any other transformation (i.e. rotation, distortion etc.) till either the vertices, initially coincident, of each four equilateral triangular faces of the octahedron form a square of the same edge length or the vertices, initially coincident, of each three square faces of the cube form an equilateral triangle of the same edge length. Both the methods create the same solid having 8 congruent **equilateral triangular** & 18 congruent **square** faces each having equal edge length. Thus solid generated is called **rhombicuboctahedron** or **small rhombicuboctahedron**.

The author had already derived the circumscribed radius of a rhombicuboctahedron by another geometrical method. Now, we will apply '**HCR's Theory of Polygon**' to derive the radius of circumscribed sphere passing through all 24 identical vertices of a rhombicuboctahedron with given edge length & then subsequently we will derive various formula to analytically compute the normal distances of equilateral triangular & square faces from the centre of rhombicuboctahedron, surface area, volume, solid angles subtended by each equilateral triangular face & each square face at the centre by using '**HCR's Theory of Polygon**', dihedral angle between each two faces meeting at any of 24 identical vertices, solid angle subtended by rhombicuboctahedron at any of its 24 identical vertices.

Derivation of radius R of spherical surface circumscribing a given rhombicuboctahedron with edge length a:

Consider a rhombicuboctahedron with edge length a & centre O such that all its 24 identical vertices lie on a sherical surface of radius R. Drop the perpendiculars OM & ON from the centre O to the centres M & N of an equilateral triangular face ABC & adjacent square face ABDE respectively. Let H_T & H_S be the normal distances of equilateral triangular face ABC & square face ABDE respectively (as shown in fig-1).

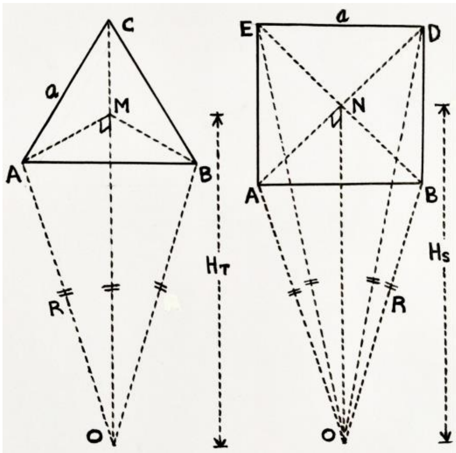

Figure 1: Equilateral triangular face ABC & square face ABDE are at normal distances H_T & H_S respectively from the centre O and $OA = OB = OC = OD = OE = R$

In right ΔAMO (see fig-1), using Pythagorean theorem, we get

$$OM = \sqrt{(OA)^2 - (AM)^2} \qquad \left(AM = \text{circumscribed radius} = \frac{a}{\sqrt{3}}\right)$$

$$H_T = \sqrt{R^2 - \left(\frac{a}{\sqrt{3}}\right)^2} = \sqrt{R^2 - \frac{a^2}{3}} = \sqrt{\frac{3R^2 - a^2}{3}} \qquad \ldots\ldots\ldots..(I)$$

Similarly, right ΔANO (see fig-1), using Pythagorean theorem, we get

$$ON = \sqrt{(OA)^2 - (AN)^2} \qquad \left(AN = \text{circumscribed radius} = \frac{a}{\sqrt{2}}\right)$$

$$H_S = \sqrt{R^2 - \left(\frac{a}{\sqrt{2}}\right)^2} = \sqrt{R^2 - \frac{a^2}{2}} = \sqrt{\frac{2R^2 - a^2}{2}} \quad \dots\dots\dots..(II)$$

We know that the solid angle (ω) subtended by any regular polygonal plane with n no. of sides each of length a at any point lying at a distance H on the vertical axis passing through the centre, is given by "**HCR's Theory of Polygon**" as follows

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2 + a^2\cot^2\frac{\pi}{n}}}\right)}$$

Now, substituting the corresponding values in above general formula i.e. number of sides $n = 3$ (for equilateral Δ), length of each side $a = AB = a$, & normal height $H = OM = H_T$, the solid angle ω_T subtended by the equilateral triangular face ABC at the centre O of rhombicuboctahedron (as shown in above fig-1), is obtained as follows

$$\omega_T = 2\pi - 2\times 3\sin^{-1}\left(\frac{2\left(\sqrt{\frac{3R^2 - a^2}{3}}\right)\sin\frac{\pi}{3}}{\sqrt{4\left(\sqrt{\frac{3R^2 - a^2}{3}}\right)^2 + a^2\cot^2\frac{\pi}{3}}}\right) = 2\pi - 6\sin^{-1}\left(\frac{2\left(\sqrt{\frac{3R^2 - a^2}{3}}\right)\frac{\sqrt{3}}{2}}{\sqrt{\frac{12R^2 - 4a^2}{3} + a^2\left(\frac{1}{\sqrt{3}}\right)^2}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{\sqrt{3R^2 - a^2}}{\sqrt{\frac{12R^2 - 4a^2 + a^2}{3}}}\right) = 2\pi - 6\sin^{-1}\left(\frac{\sqrt{3R^2 - a^2}}{\sqrt{4R^2 - a^2}}\right) = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3R^2 - a^2}{4R^2 - a^2}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\sqrt{\frac{a^2\left(\frac{3R^2}{a^2} - 1\right)}{a^2\left(\frac{4R^2}{a^2} - 1\right)}}\right) = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{\frac{3R^2}{a^2} - 1}{\frac{4R^2}{a^2} - 1}}\right) = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3\left(\frac{R}{a}\right)^2 - 1}{4\left(\frac{R}{a}\right)^2 - 1}}\right)$$

Now, for ease of calculation, let's assume $\frac{R}{a} = x$ $(\forall\ x > 1)$ then we get

$$\omega_T = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right) \quad \dots\dots\dots\dots.(III)$$

Similarly, substituting the corresponding values in above general formula i.e. number of sides $n = 4$ (for square), length of each side $a = AB = a$, & normal height $H = ON = H_S$, the solid angle ω_S subtended by the square face ABDE at the centre O of rhombicuboctahedron (as shown in above fig-1), is obtained as follows

$$\omega_S = 2\pi - 2\times 4\sin^{-1}\left(\frac{2\left(\sqrt{\frac{2R^2 - a^2}{2}}\right)\sin\frac{\pi}{4}}{\sqrt{4\left(\sqrt{\frac{2R^2 - a^2}{2}}\right)^2 + a^2\cot^2\frac{\pi}{4}}}\right) = 2\pi - 8\sin^{-1}\left(\frac{2\left(\sqrt{\frac{2R^2 - a^2}{2}}\right)\frac{1}{\sqrt{2}}}{\sqrt{\frac{8R^2 - 4a^2}{2} + a^2(1)^2}}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\frac{\sqrt{2R^2 - a^2}}{\sqrt{\frac{8R^2 - 4a^2 + 2a^2}{2}}}\right) = 2\pi - 8\sin^{-1}\left(\frac{\sqrt{2R^2 - a^2}}{\sqrt{4R^2 - a^2}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2R^2 - a^2}{4R^2 - a^2}}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\sqrt{\frac{a^2\left(\frac{2R^2}{a^2} - 1\right)}{a^2\left(\frac{4R^2}{a^2} - 1\right)}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{\frac{2R^2}{a^2} - 1}{\frac{4R^2}{a^2} - 1}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2\left(\frac{R}{a}\right)^2 - 1}{4\left(\frac{R}{a}\right)^2 - 1}}\right)$$

Now, for ease of calculation, let's assume $\frac{R}{a} = x \quad (\forall\ x > 1)$ then we get

$$\omega_S = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2 - 1}{4x^2 - 1}}\right) \qquad \dots\dots\dots\dots.(IV)$$

But we know that solid angle, subtended by any closed surface at any point inside it, is always equal to 4π sr (this fact has also been mathematically proven by the author). Since a rhombicuboctahedron is a closed surface consisting of 8 congruent equilateral triangular faces & 18 congruent square faces therefore the solid angle subtended by all 26 faces at the centre O of rhombicuboctahedron must be equal to 4π sr as follows

$$8(\omega_T) + 18(\omega_S) = 4\pi$$

Substituting the corresponding values of solid angles ω_T & ω_S from above eq(III) & (IV) as follows

$$8\left(2\pi - 6\sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right)\right) + 18\left(2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2 - 1}{4x^2 - 1}}\right)\right) = 4\pi$$

$$16\pi - 48\sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right) + 36\pi - 144\sin^{-1}\left(\sqrt{\frac{2x^2 - 1}{4x^2 - 1}}\right) = 4\pi$$

$$48\pi - 48\sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right) - 144\sin^{-1}\left(\sqrt{\frac{2x^2 - 1}{4x^2 - 1}}\right) = 0$$

$$\pi - \sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right) - 3\sin^{-1}\left(\sqrt{\frac{2x^2 - 1}{4x^2 - 1}}\right) = 0$$

$$\pi - \sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right) - \sin^{-1}\left(\sqrt{\frac{2x^2 - 1}{4x^2 - 1}}\right) = 2\sin^{-1}\left(\sqrt{\frac{2x^2 - 1}{4x^2 - 1}}\right)$$

$$\pi - \left(\sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right) + \sin^{-1}\left(\sqrt{\frac{2x^2 - 1}{4x^2 - 1}}\right)\right) = 2\sin^{-1}\left(\sqrt{\frac{2x^2 - 1}{4x^2 - 1}}\right)$$

Using following formula,
$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right)$ & $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1 - x^2}\right)$ $\forall\ |x|, |y| \in [0,1]$

$$\pi-\left(\sin^{-1}\left(\sqrt{\frac{3x^2-1}{4x^2-1}}\sqrt{1-\frac{2x^2-1}{4x^2-1}}+\sqrt{\frac{2x^2-1}{4x^2-1}}\sqrt{1-\frac{3x^2-1}{4x^2-1}}\right)\right)=\sin^{-1}\left(2\sqrt{\frac{2x^2-1}{4x^2-1}}\sqrt{1-\frac{2x^2-1}{4x^2-1}}\right)$$

$$\pi-\left(\sin^{-1}\left(\sqrt{\frac{3x^2-1}{4x^2-1}}\sqrt{\frac{2x^2}{4x^2-1}}+\sqrt{\frac{2x^2-1}{4x^2-1}}\sqrt{\frac{x^2}{4x^2-1}}\right)\right)=\sin^{-1}\left(2\sqrt{\frac{2x^2-1}{4x^2-1}}\sqrt{\frac{2x^2}{4x^2-1}}\right)$$

$$\pi-\sin^{-1}\left(\frac{x\sqrt{2}\sqrt{3x^2-1}}{4x^2-1}+\frac{x\sqrt{2x^2-1}}{4x^2-1}\right)=\sin^{-1}\left(\frac{2x\sqrt{2}\sqrt{2x^2-1}}{4x^2-1}\right)$$

$$\pi-\sin^{-1}\left(\frac{x\left(\sqrt{6x^2-2}+\sqrt{2x^2-1}\right)}{4x^2-1}\right)=\sin^{-1}\left(\frac{2x\sqrt{2}\sqrt{2x^2-1}}{4x^2-1}\right)$$

Taking Sine on both the sides, we get

$$\sin\left(\pi-\sin^{-1}\left(\frac{x\left(\sqrt{6x^2-2}+\sqrt{2x^2-1}\right)}{4x^2-1}\right)\right)=\sin\left(\sin^{-1}\left(\frac{2x\sqrt{2}\sqrt{2x^2-1}}{4x^2-1}\right)\right)$$

$$\sin\left(\sin^{-1}\left(\frac{x\left(\sqrt{6x^2-2}+\sqrt{2x^2-1}\right)}{4x^2-1}\right)\right)=\frac{2x\sqrt{2}\sqrt{2x^2-1}}{4x^2-1}$$

$$\frac{x\left(\sqrt{6x^2-2}+\sqrt{2x^2-1}\right)}{4x^2-1}=\frac{2x\sqrt{2}\sqrt{2x^2-1}}{4x^2-1}$$

$$\frac{x\left(\sqrt{6x^2-2}+\sqrt{2x^2-1}\right)}{4x^2-1}-\frac{2x\sqrt{2}\sqrt{2x^2-1}}{4x^2-1}=0$$

$$\frac{x\left(\sqrt{6x^2-2}+\sqrt{2x^2-1}-2\sqrt{2}\sqrt{2x^2-1}\right)}{4x^2-1}=0$$

$$\sqrt{6x^2-2}+\sqrt{2x^2-1}-2\sqrt{2}\sqrt{2x^2-1}=0 \qquad (\text{since, } x\neq 0,\ 4x^2-1\neq 0)$$

$$\sqrt{6x^2-2}-\left(2\sqrt{2}-1\right)\sqrt{2x^2-1}=0$$

$$\sqrt{6x^2-2}=\left(2\sqrt{2}-1\right)\sqrt{2x^2-1}$$

$$\left(\sqrt{6x^2-2}\right)^2=\left(\left(2\sqrt{2}-1\right)\sqrt{2x^2-1}\right)^2$$

$$6x^2-2=\left(9-4\sqrt{2}\right)(2x^2-1)$$

$$6x^2-2=\left(18-8\sqrt{2}\right)x^2-9+4\sqrt{2}$$

$$\left(18-8\sqrt{2}\right)x^2-6x^2=-2+9-4\sqrt{2}$$

$$4\left(3-2\sqrt{2}\right)x^2=7-4\sqrt{2}$$

$$x^2=\frac{7-4\sqrt{2}}{4\left(3-2\sqrt{2}\right)}$$

$$x^2=\frac{\left(7-4\sqrt{2}\right)\left(3+2\sqrt{2}\right)}{4\left(3-2\sqrt{2}\right)\left(3+2\sqrt{2}\right)}$$

$$x^2 = \frac{21 - 12\sqrt{2} + 14\sqrt{2} - 16}{4\left(3^2 - \left(2\sqrt{2}\right)^2\right)}$$

$$x^2 = \frac{5 + 2\sqrt{2}}{4\,(9 - 8)} = \frac{5 + 2\sqrt{2}}{4}$$

$$x = \sqrt{\frac{5 + 2\sqrt{2}}{4}} = \frac{\sqrt{\mathbf{5 + 2\sqrt{2}}}}{\mathbf{2}} > 1 \qquad \left(\text{Condition is satisfied i.e. } x = \frac{R}{a} > 1\right)$$

$$\Rightarrow \quad \frac{R}{a} = x = \frac{\sqrt{5 + 2\sqrt{2}}}{2} \quad \Rightarrow \quad R = \frac{a\sqrt{5 + 2\sqrt{2}}}{2}$$

Therefore, the **radius (R) of spherical surface passing through all 24 identical vertices of a given rhombicuboctahedron with edge length a** is given as follows

$$\boldsymbol{R = \frac{a}{2}\sqrt{5 + 2\sqrt{2}} \approx 1.398966326a} \qquad \ldots\ldots\ldots\ldots(\mathbf{1})$$

Normal distances H_T & H_S of equilateral triangular & square faces from the centre of rhombicuboctahedron: Substituting the value of radius R of circumscribed sphere in the eq(I) (as derived above from fig-1), the normal distance $OM = H_T$ of equilateral triangular face ABC from the centre O is

$$H_T = \sqrt{\frac{3R^2 - a^2}{3}} = \sqrt{\frac{3\left(\frac{a\sqrt{5 + 2\sqrt{2}}}{2}\right)^2 - a^2}{3}} = \sqrt{\frac{\frac{3a^2(5 + 2\sqrt{2})}{4} - a^2}{3}} = \sqrt{\frac{a^2(11 + 6\sqrt{2})}{12}} = \sqrt{\frac{a^2(3 + \sqrt{2})^2}{12}}$$

$$H_T = \frac{a(3 + \sqrt{2})}{2\sqrt{3}}$$

Similarly, substituting the value of radius R of circumscribed sphere in the eq(II) (as derived above from fig-1), the normal distance $ON = H_S$ of square face ABDE from the centre O is

$$H_S = \sqrt{\frac{2R^2 - a^2}{2}} = \sqrt{\frac{2\left(\frac{a\sqrt{5 + 2\sqrt{2}}}{2}\right)^2 - a^2}{2}} = \sqrt{\frac{\frac{2a^2(5 + 2\sqrt{2})}{4} - a^2}{2}} = \sqrt{\frac{a^2(3 + 2\sqrt{2})}{4}} = \sqrt{\frac{a^2(1 + \sqrt{2})^2}{4}}$$

$$H_S = \frac{a(1 + \sqrt{2})}{2}$$

Hence, the **normal distances H_T & H_S of equilateral triangular & square faces respectively from the centre of rhombicuboctahedron with edge length a**, are given as follows

$$\boldsymbol{H_T = \frac{a(3 + \sqrt{2})}{2\sqrt{3}} \approx 1.274273694\,a}\ , \quad \& \quad \boldsymbol{H_S = \frac{a(1 + \sqrt{2})}{2} \approx 1.207106781\,a} \qquad \ldots\ldots\ldots\ldots(\mathbf{2})$$

It is clear from above values of normal distances that the equilateral triangular faces are farther from the centre while square faces are closer to the centre. For finite value of edge length $a \Rightarrow \boldsymbol{H_S < H_T < R}$

Surface Area (A_s) of rhombicuboctahedron: The surface of a rhombicuboctahedron consists of 8 congruent equilateral triangular faces each of side a & 18 congruent square faces each with side a. Therefore, the (total) surface area of rhombicuboctahedron is given as follows

$$A_s = 8(\text{Area of equilateral triangular face}) + 18(\text{Area of square face})$$

$$= 8\left(\frac{\sqrt{3}}{4}a^2\right) + 18(a^2) = 2a^2\sqrt{3} + 18a^2 = 2a^2(9 + \sqrt{3})$$

$$\therefore \textbf{Surface area},\ \boldsymbol{A_s = 2a^2(9 + \sqrt{3}) \approx 21.46410162\ a^2}$$

Volume (V) of truncated rhombic dodecahedron: The surface of a rhombicuboctahedron consists of 8 congruent equilateral triangular faces each of side a & 18 congruent square faces each with side a. Thus a solid rhombicuboctahedron can assumed to consisting of 8 congruent right pyramids with equilateral triangular base with side a & normal height H_T and 18 congruent right pyramids with square base of side a & normal height H_S (as shown in above fig-1). Therefore, the volume of rhombicuboctahedron is given as

$$V = 8(\text{Volume of equilateral triangular right pyramid}) + 18(\text{Volume of square right pyramid})$$

$$= 8\left(\frac{1}{3}\left(\frac{\sqrt{3}}{4}a^2\right)\cdot\frac{a(3+\sqrt{2})}{2\sqrt{3}}\right) + 18\left(\frac{1}{3}a^2\cdot\frac{a(1+\sqrt{2})}{2}\right)$$

$$= \frac{a^3(3+\sqrt{2})}{3} + 3a^3(1+\sqrt{2})$$

$$= \frac{a^3(3+\sqrt{2}) + 9a^3(1+\sqrt{2})}{3} = \frac{a^3(12+10\sqrt{2})}{3} = \frac{2a^3(6+5\sqrt{2})}{3}$$

$$\therefore \textbf{Volume},\ \boldsymbol{V = \frac{2a^3(6+5\sqrt{2})}{3} \approx 8.714045208\ a^3}$$

Mean radius (R_m) of rhombicuboctahedron: It is the radius of the sphere having a volume equal to that of a given rhombicuboctahedron with edge lengths a. It is computed as follows

$$\text{volume of sphere with mean radius } \mathrm{R_m} = \text{volume of rhombicuboctahedron with edge } a$$

$$\frac{4}{3}\pi(R_m)^3 = \frac{2a^3(6+5\sqrt{2})}{3}$$

$$(R_m)^3 = \frac{a^3(6+5\sqrt{2})}{2\pi}$$

$$R_m = \left(\frac{a^3(6+5\sqrt{2})}{2\pi}\right)^{1/3}$$

$$R_m = a\left(\frac{6+5\sqrt{2}}{2\pi}\right)^{1/3}$$

$$\therefore \textbf{Mean radius},\ \boldsymbol{R_m = a\left(\frac{6+5\sqrt{2}}{2\pi}\right)^{1/3} \approx 1.276567352\ a} \qquad \ldots\ldots\ldots\ldots(3)$$

It's worth noticing that for a finite value of edge length $a \Rightarrow \boldsymbol{H_S < H_T < R_m < R}$

Dihedral angle between any two faces meeting at a vertex of rhombicuboctahedron:

A rhombicuboctahedron has 24 identical vertices at each of which one equilateral triangular & three square faces meet together. Consider a vertex A at which an equilateral triangular face ABC and three square faces ABDE, AFGH & AHIJ meet together (as shown in fig-2). Therefore, we will consider each pair of two faces meeting at the vertex A & find the **dihedral angle measured internally** between each two faces. For ease of understanding we would like to use following symbols for different dihedral angles

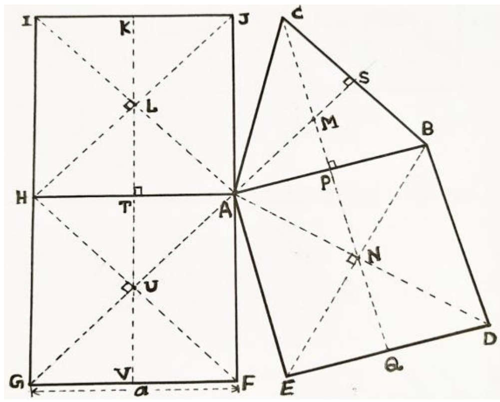

Figure 2: One equilateral triangular & three square faces meeting at vertex A, are flattened onto the plane of paper (projected view of faces)

$\theta_{TS} \rightarrow$ Dihedral angle between equilateral triangular & square faces with common side. The subscript **T** stands for **Triangle** & **S** stands for **Side common**

$\theta_{TV} \rightarrow$ Dihedral angle between equilateral triangular & square faces with common vertex. Subscript **T** stands for **Triangle** & **V** stands for **Vertex common**

$\theta_{SS} \rightarrow$ Dihedral angle between two square faces with common side. The subscript **T** stands for **Triangle** & **S** stands for **Side common**

$\theta_{SV} \rightarrow$ Dihedral angle between two square faces with common vertex only. The subscript **T** stands for **Triangle** & **S** stands for **Vertex common**

The above symbols show that there are four different types of dihedral angles (all are measured internally) between all possible distinct pairs of the faces meeting at a vertex of rhombicuboctahedron.

Consider an equilateral triangular face ABC and a square face ABDE having a common side AB & meeting each other at the vertex A (see above fig-2) such that they are inclined at a dihedral angle $\boldsymbol{\theta_{TS}}$. The line CP shows the altitude of equilateral triangular face ABC (see fig-2 above) & line PQ shows the line-segment joining the mid-points of opposite sides AB & DE of square face ABDE (see fig-2 above). Now, drop the perpendiculars OM & ON from the centre O of rhombicuboctahedron to the centres M & N of triangular & square faces respectively (as shown in fig-3).

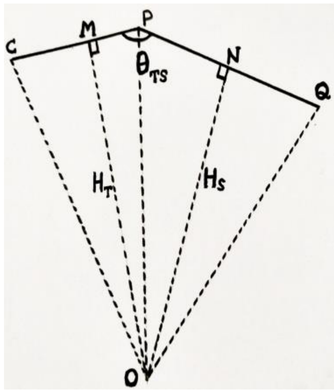

Figure 3: Dihedral angle $\angle CPQ = \theta_{TS}$ between equi. triangular & square faces shown by the lines CP & PQ ⊥ to the plane of paper

In right ΔOMP (see fig-3),

$$\tan\angle MPO = \frac{OM}{PM} = \frac{H_T}{\left(\frac{PC}{3}\right)} = \frac{\left(\frac{a\left(3+\sqrt{2}\right)}{2\sqrt{3}}\right)}{\left(\frac{\frac{a\sqrt{3}}{2}}{3}\right)} = 3+\sqrt{2} \qquad \left(\text{since, } \ \mathrm{PC} = a\sin 60^o = \frac{a\sqrt{3}}{2}\right)$$

$$\angle MPO = \tan^{-1}\left(3+\sqrt{2}\right)$$

In right ΔONP (see fig-3),

$$\tan\angle NPO = \frac{ON}{PN} = \frac{H_S}{\left(\frac{PQ}{2}\right)} = \frac{\left(\frac{a\left(1+\sqrt{2}\right)}{2}\right)}{\left(\frac{a}{2}\right)} = 1+\sqrt{2} \qquad (\text{since, } \ \mathrm{PQ} = \mathrm{AE} = a)$$

$$\angle NPO = \tan^{-1}\left(1+\sqrt{2}\right)$$

$$\Rightarrow \angle CPQ = \angle MPO + \angle NPO$$

$$\theta_{TS} = \tan^{-1}(3+\sqrt{2}) + \tan^{-1}(1+\sqrt{2})$$

$$= \pi + \tan^{-1}\left(\frac{3+\sqrt{2}+1+\sqrt{2}}{1-(3+\sqrt{2})(1+\sqrt{2})}\right) \qquad \left(\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)\ \forall\, xy > 1\right)$$

$$= \pi + \tan^{-1}\left(\frac{4+2\sqrt{2}}{-4-4\sqrt{2}}\right) = \pi + \tan^{-1}\left(\frac{2\sqrt{2}(\sqrt{2}+1)}{-4(\sqrt{2}+1)}\right) = \pi + \tan^{-1}\left(\frac{1}{-\sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

Hence, the **dihedral angle θ_{TS} (measured internally) between any two adjacent equilateral triangular & square faces of a rhombicuboctahedron**, is given as follows

$$\boldsymbol{\theta_{TS} = \pi - \tan^{-1}\frac{1}{\sqrt{2}} \approx 144.73^o} \qquad \dots\dots\dots\dots(4)$$

Consider an equilateral triangular face ABC and a square face AFGH meeting each other at the vertex A & having no side common (see above fig-2) such that they are inclined at a dihedral angle $\boldsymbol{\theta_{TV}}$. The line AS shows the altitude of equilateral triangular face ABC (see fig-2 above) & line AG shows the diagonal of square face AFGH (see fig-2 above). Now, drop the perpendiculars OM & OU from the centre O of rhombicuboctahedron to the centres M & U of triangular & square faces respectively (as shown in fig-4).

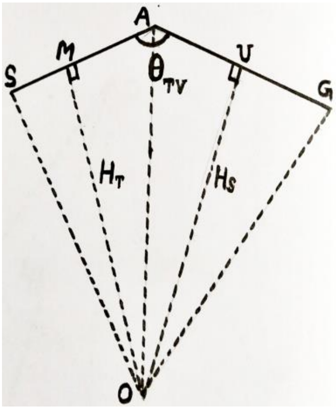

Figure 4: Dihedral angle $\angle SAG = \theta_{TV}$ between equi. triangular & square faces shown by the lines AS & AG ⊥ to the plane of paper

In right ΔOMA (see fig-4),

$$\tan\angle MAO = \frac{OM}{AM} = \frac{H_T}{\left(\frac{2AS}{3}\right)} = \frac{\left(\frac{a(3+\sqrt{2})}{2\sqrt{3}}\right)}{\left(\frac{2\cdot\frac{a\sqrt{3}}{2}}{3}\right)} = \frac{3+\sqrt{2}}{2} \qquad \left(\text{since, } \ \text{AS} = a\sin 60^o = \frac{a\sqrt{3}}{2}\right)$$

$$\angle MAO = \tan^{-1}\left(\frac{3+\sqrt{2}}{2}\right)$$

In right ΔOUA (see fig-4),

$$\tan\angle UAO = \frac{OU}{AU} = \frac{H_S}{\left(\frac{AG}{2}\right)} = \frac{\left(\frac{a(1+\sqrt{2})}{2}\right)}{\left(\frac{a\sqrt{2}}{2}\right)} = \frac{1+\sqrt{2}}{\sqrt{2}} \qquad \left(\text{since, } \ \text{AG} = \text{diagonal} = a\sqrt{2}\right)$$

$$\angle UAO = \tan^{-1}\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)$$

$$\Rightarrow \angle SAG = \angle MAO + \angle UAO$$

$$\theta_{TV} = \tan^{-1}\left(\frac{3+\sqrt{2}}{2}\right) + \tan^{-1}\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)$$

$$= \pi + \tan^{-1}\left(\frac{\frac{3+\sqrt{2}}{2}+\frac{1+\sqrt{2}}{\sqrt{2}}}{1-\left(\frac{3+\sqrt{2}}{2}\right)\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)}\right) \qquad \left(\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)\ \forall\, xy > 1\right)$$

$$= \pi + \tan^{-1}\left(\frac{\frac{5+2\sqrt{2}}{2}}{1-\frac{5+4\sqrt{2}}{2\sqrt{2}}}\right) = \pi + \tan^{-1}\left(\frac{\sqrt{2}(5+2\sqrt{2})}{-(5+2\sqrt{2})}\right) = \pi + \tan^{-1}(-\sqrt{2}) = \pi - \tan^{-1}(\sqrt{2})$$

Hence, the **dihedral angle θ_{TV} (measured internally) between equilateral triangular & square faces (with no side common) meeting at a vertex of rhombicuboctahedron**, is given as follows

$$\boldsymbol{\theta_{TV} = \pi - \tan^{-1}\sqrt{2} \approx 125.26^o} \qquad \ldots\ldots\ldots\ldots(5)$$

Consider two congruent square faces AFGH & AHIJ having a common side AH & meeting each other at the vertex A (see above fig-2) such that they are inclined at a dihedral angle θ_{SS}. The line TK joins the mid-points of sides AH & IJ of square face AHIJ & the line TV joins the mid-points of sides AH & FG of square face AFGH. Drop the perpendiculars OL & OU from the centre O to of rhombicuboctahedron to the centres L & U of square faces AHIJ & AFGH respectively (see fig-5).

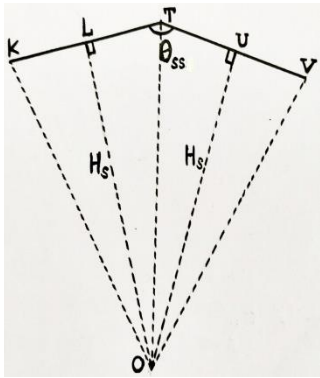

Figure 5: Dihedral angle $\angle KTV = \theta_{SS}$ between two square faces with a common side AH shown by lines TK & TV ⊥ to the plane of paper

In right ΔOLT (see fig-5),

$$\tan\angle LTO = \frac{OL}{TL} = \frac{H_S}{\left(\frac{AJ}{2}\right)} = \frac{\left(\frac{a(1+\sqrt{2})}{2}\right)}{\left(\frac{a}{2}\right)} = 1+\sqrt{2} \qquad (\text{since, AJ} = \text{side of square} = a\,)$$

$$\angle LTO = \tan^{-1}(1+\sqrt{2})$$

$$\Rightarrow \angle UTO = \angle LTO = \tan^{-1}(1+\sqrt{2})$$

$$\therefore \angle KTV = \angle LTO + \angle UTO$$

$$\theta_{RR} = \tan^{-1}(1+\sqrt{2}) + \tan^{-1}(1+\sqrt{2})$$

$$= 2\tan^{-1}(1+\sqrt{2})$$

$$= \pi + \tan^{-1}\left(\frac{2(1+\sqrt{2})}{1-(1+\sqrt{2})^2}\right)$$

$$= \pi + \tan^{-1}\left(\frac{2+2\sqrt{2}}{-2-2\sqrt{2}}\right) = \pi + \tan^{-1}(-1) = \pi + \left(-\frac{\pi}{4}\right) = \frac{3\pi}{4}$$

Hence, the **dihedral angle θ_{SS} between any two adjacent square faces of rhombicuboctahedron**, is given as follows

$$\boldsymbol{\theta_{SS} = \frac{3\pi}{4} = 135^o} \qquad \ldots\ldots\ldots\ldots(6)$$

Consider two congruent square faces ABDE & AHIJ meeting each other at the vertex A (with no side common) (see left diagram in fig-6) such that they are inclined at a dihedral angle $\boldsymbol{\theta_{SV}}$. It is worth noticing that the diagonals AD & AI of square faces ABDE & AHIJ respectively are not collinear. Therefore draw a straight line from common vertex A which intersect the diagonals BE & HJ at the points N' & L' respectively such that the angle between AN' & side AB is θ_P and similarly angle between AL' & side AJ is θ_P.

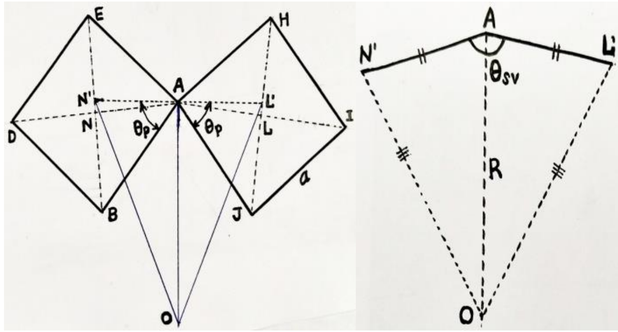

Figure 6: Dihedral angle $\angle N'AL' = \theta_{SV}$ between two square faces ABDE & AHIJ meeting at the vertex A & having no side common (left diagram is top-view). $ON = H_S$

Angle θ_P is computed by **HCR's Cosine Formula** as follows

$$\boldsymbol{\theta_P = \cos^{-1}\left(\frac{\sqrt{1532 - 368\sqrt{2} + 16\sqrt{34\left(71 - 8\sqrt{2}\right)}}}{68}\right) \approx 52.26^o}$$

In right $\Delta ANN'$ (see left diagram in fig-6),

$$\tan\angle NAN' = \frac{NN'}{AN} \ \Rightarrow \tan\left(\frac{\pi}{4} - \theta_P\right) = \frac{NN'}{\frac{a\sqrt{2}}{2}} \ \Rightarrow \boldsymbol{NN'} = \frac{a}{\sqrt{2}}\tan\left(\frac{\pi}{4} - \theta_P\right) \ \ \& \ \boldsymbol{AN'} = \frac{a}{\sqrt{2}}\sec\left(\frac{\pi}{4} - \theta_P\right)$$

Using Pythagorean theorem in right $\Delta ONN'$ (see left diagram in fig-6), as follows

$$(\boldsymbol{ON'})^2 = (ON)^2 + (NN')^2 = \left(\frac{a\left(1+\sqrt{2}\right)}{2}\right)^2 + \left(\frac{a}{\sqrt{2}}\tan\left(\frac{\pi}{4} - \theta_P\right)\right)^2 = \frac{a^2\left(3 + 2\sqrt{2}\right)}{4} + \frac{a^2}{2}\tan^2\left(\frac{\pi}{4} - \theta_P\right)$$

Now, using Cosine rule in $\Delta AN'O$ (see right diagram in fig-6) as follows

$$\cos\angle N'AO = \frac{(AO)^2 + (AN')^2 - (ON')^2}{2(AO)(AN')}$$

$$\cos\frac{\theta_{SV}}{2} = \frac{\left(\frac{a}{2}\sqrt{5+2\sqrt{2}}\right)^2 + \left(\frac{a}{\sqrt{2}}\sec\left(\frac{\pi}{4} - \theta_P\right)\right)^2 - \left(\frac{a^2\left(3+2\sqrt{2}\right)}{4} + \frac{a^2}{2}\tan^2\left(\frac{\pi}{4} - \theta_P\right)\right)}{2\left(\frac{a}{2}\sqrt{5+2\sqrt{2}}\right)\left(\frac{a}{\sqrt{2}}\sec\left(\frac{\pi}{4} - \theta_P\right)\right)}$$

$$\cos\frac{\theta_{SV}}{2} = \frac{\frac{a^2\left(5+2\sqrt{2}\right)}{4} + \frac{a^2}{2}\sec^2\left(\frac{\pi}{4} - \theta_P\right) - \frac{a^2\left(3+2\sqrt{2}\right)}{4} - \frac{a^2}{2}\tan^2\left(\frac{\pi}{4} - \theta_P\right)}{\frac{a^2}{\sqrt{2}}\sqrt{5+2\sqrt{2}}\sec\left(\frac{\pi}{4} - \theta_P\right)}$$

$$\cos\frac{\theta_{SV}}{2} = \frac{\frac{5+2\sqrt{2}-3-2\sqrt{2}}{4} + \frac{1}{2}\left(\sec^2\left(\frac{\pi}{4} - \theta_P\right) - \tan^2\left(\frac{\pi}{4} - \theta_P\right)\right)}{\frac{1}{\sqrt{2}}\sqrt{5+2\sqrt{2}}\sec\left(\frac{\pi}{4} - \theta_P\right)}$$

$$\cos\frac{\theta_{SV}}{2} = \frac{\frac{1}{2} + \frac{1}{2}}{\frac{1}{\sqrt{2}}\sqrt{5+2\sqrt{2}}\sec\left(\frac{\pi}{4} - \theta_P\right)} = \frac{\sqrt{2}\cos\left(\frac{\pi}{4} - \theta_P\right)}{\sqrt{5+2\sqrt{2}}} = \frac{\sqrt{2}\cos\left(\frac{\pi}{4} - \theta_P\right)\sqrt{5-2\sqrt{2}}}{\sqrt{5+2\sqrt{2}}\sqrt{5-2\sqrt{2}}}$$

$$\cos\frac{\theta_{SV}}{2} = \sqrt{\frac{10-4\sqrt{2}}{17}}\cos\left(\frac{\pi}{4}-\theta_P\right) \quad \Rightarrow \quad \theta_{SV} = 2\cos^{-1}\left(\sqrt{\frac{10-4\sqrt{2}}{17}}\cos\left(\frac{\pi}{4}-\theta_P\right)\right)$$

Hence, the **dihedral angle θ_{SV} between any two square faces, meeting at a vertex with no common side, of rhombicuboctahedron**, is given as follows

$$\boldsymbol{\theta_{SV} = 2\cos^{-1}\left(\sqrt{\frac{10-4\sqrt{2}}{17}}\cos\left(\frac{\pi}{4}-\theta_P\right)\right) \approx 119.815^o} \quad \ldots\ldots\ldots\ldots(7)$$

Where, $\boldsymbol{\theta_P \approx 52.26^o}$ (as given above by HCR's Cosine Formula)

Solid angles ω_T & ω_S subtended by equilateral triangular & square faces respectively at the centre of rhombicuboctahedron:

As we have already derived the formula of solid angle, subtended by an equilateral triangular face at the centre of rhombicuboctahedron, given by above equ(III) as follows

$$\omega_T = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3x^2-1}{4x^2-1}}\right)$$

Substituting the value of x (as derived above using HCR's Theory of Polygon) in above formula, we get

$$\omega_T = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3\left(\frac{\sqrt{5+2\sqrt{2}}}{2}\right)^2-1}{4\left(\frac{\sqrt{5+2\sqrt{2}}}{2}\right)^2-1}}\right) = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{15+6\sqrt{2}-4}{20+8\sqrt{2}-4}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\sqrt{\frac{11+6\sqrt{2}}{16+8\sqrt{2}}}\right) = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{(11+6\sqrt{2})(2-\sqrt{2})}{8(2+\sqrt{2})(2-\sqrt{2})}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\sqrt{\frac{22+12\sqrt{2}-11\sqrt{2}-12}{8(4-2)}}\right) = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{10+\sqrt{2}}{16}}\right) = 2\pi - 6\sin^{-1}\left(\frac{\sqrt{10+\sqrt{2}}}{4}\right)$$

Hence, the **solid angle ω_T subtended by any of 8 congruent equilateral triangular faces at the centre of a rhombicuboctahedron**, is given as follows

$$\boldsymbol{\omega_T = 2\pi - 6\sin^{-1}\left(\frac{\sqrt{10+\sqrt{2}}}{4}\right)\ \text{sr} \approx 0.24801961\ \text{sr}} \quad \ldots\ldots\ldots\ldots(8)$$

Similarly, solid angle subtended by an equilateral triangular face at the centre of rhombicuboctahedron is given by above equ(IV) as follows

$$\omega_S = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right)$$

Substituting the value of x (as derived above using HCR's Theory of Polygon) in above formula, we get

$$\omega_S = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2\left(\frac{\sqrt{5+2\sqrt{2}}}{2}\right)^2 - 1}{4\left(\frac{\sqrt{5+2\sqrt{2}}}{2}\right)^2 - 1}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{10+4\sqrt{2}-4}{20+8\sqrt{2}-4}}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\sqrt{\frac{6+4\sqrt{2}}{16+8\sqrt{2}}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{(6+4\sqrt{2})(2-\sqrt{2})}{8(2+\sqrt{2})(2-\sqrt{2})}}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\sqrt{\frac{12+8\sqrt{2}-6\sqrt{2}-8}{8(4-2)}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{4+2\sqrt{2}}{16}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{\sqrt{2}+1}{4\sqrt{2}}}\right)$$

$$= 4\left(\frac{\pi}{2} - 2\sin^{-1}\left(\sqrt{\frac{\sqrt{2}+1}{4\sqrt{2}}}\right)\right) = 4\left(\frac{\pi}{2} - \sin^{-1}\left(2\sqrt{\frac{\sqrt{2}+1}{4\sqrt{2}}}\sqrt{1-\frac{\sqrt{2}+1}{4\sqrt{2}}}\right)\right)$$

$$= 4\left(\frac{\pi}{2} - \sin^{-1}\left(\sqrt{\frac{5+2\sqrt{2}}{8}}\right)\right) = 4\cos^{-1}\left(\sqrt{\frac{5+2\sqrt{2}}{8}}\right) = 4\sin^{-1}\left(\sqrt{1-\frac{5+2\sqrt{2}}{8}}\right) = 4\sin^{-1}\left(\sqrt{\frac{3-2\sqrt{2}}{8}}\right)$$

$$= 4\sin^{-1}\left(\sqrt{\frac{6-4\sqrt{2}}{16}}\right) = 4\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})^2}{4^2}}\right) = 4\sin^{-1}\left(\frac{2-\sqrt{2}}{4}\right)$$

Hence, the **solid angle ω_S subtended by any of 18 congruent square faces at the centre of a rhombicuboctahedron**, is given as follows

$$\boldsymbol{\omega_S = 4\sin^{-1}\left(\frac{2-\sqrt{2}}{4}\right)\ \mathrm{sr} \approx 0.587900762\ \mathrm{sr}} \qquad \ldots\ldots\ldots\ldots(9)$$

Solid angle subtended by rhombicuboctahedron at any of its 24 identical vertices:

Consider any of 24 identical vertices say vertex P of rhombicuboctahedron with edge length a. Join the end points A, B, C & D of the edges AP, BP, CP & DP, meeting at vertex P, to get a (plane) trapezium ABCD of sides $AB = a,\ BC = CD = AD = a\sqrt{2}$ (as shown in fig-7).

Join the foot point Q of perpendicular PQ drawn from vertex P to the plane of trapezium ABCD, to the vertices A, B, C & D. Drop the perpendiculars MN & QE from midpoint M & foot point Q to the sides CD & BC respectively of trapezium ABCD (as shown in the fig-8).

We have found out that dihedral angle between equilateral triangular & square faces is $\angle MPN = \pi - \tan^{-1}\sqrt{2}$.

The perpendicular PQ dropped from the vertex

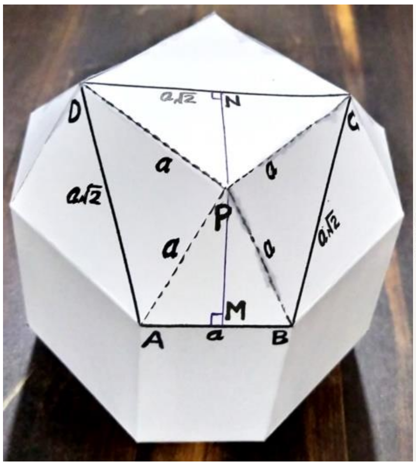

Figure 7: A trapezium ABCD is formed by joining the endpoints A, B, C & D of edges AP, BP, CP & DP meeting at the vertex P of given polyhedron

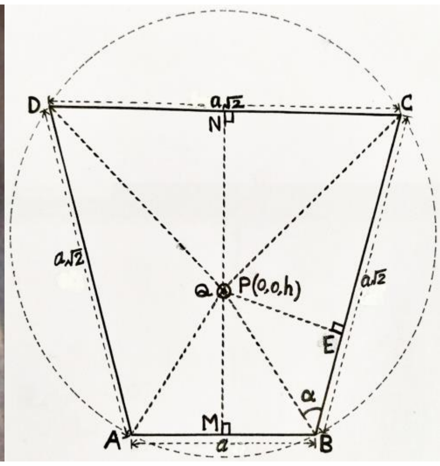

Figure 8: Point Q is the foot of perpendicular PQ drawn from the vertex P to the plane of trapezium ABCD. Point P is lying at a normal height h from the point foot Q (⊥ to the plane of paper).

P to the plane of trapezium ABCD will fall at the point Q lying on the line MN (as shown in fig-9).

In ΔMPN (see fig-9), using cosine formula as follows

$$\cos\angle MPN=\frac{(PM)^2+(PN)^2-(MN)^2}{2(PM)(PN)}$$

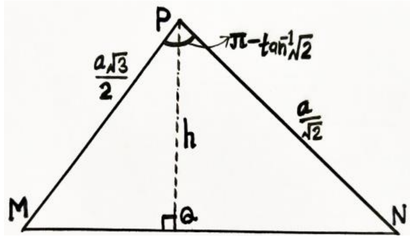

Figure 9: Perpendicular PQ dropped from vertex P to the plane of trapezium ABCD falls at the point Q on the line MN

$$\Rightarrow \cos(\pi-\tan^{-1}\sqrt{2})=\frac{\left(\frac{a\sqrt{3}}{2}\right)^2+\left(\frac{a}{\sqrt{2}}\right)^2-(MN)^2}{2\left(\frac{a\sqrt{3}}{2}\right)\left(\frac{a}{\sqrt{2}}\right)}$$

$$-\cos(\tan^{-1}\sqrt{2})=\frac{\frac{3a^2}{4}+\frac{a^2}{2}-MN^2}{a^2\sqrt{\frac{3}{2}}}$$

$$-\cos\left(\cos^{-1}\frac{1}{\sqrt{3}}\right)=\frac{\frac{5a^2}{4}-MN^2}{a^2\sqrt{\frac{3}{2}}}\quad\Rightarrow -\frac{1}{\sqrt{3}}=\frac{\frac{5a^2}{4}-MN^2}{a^2\sqrt{\frac{3}{2}}}\quad\Rightarrow -a^2\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}=\frac{5a^2}{4}-MN^2$$

$$\Rightarrow\ MN^2=\frac{5a^2}{4}+\frac{a^2}{\sqrt{2}}\ \Rightarrow\ MN^2=\frac{a^2(5+2\sqrt{2})}{4}\quad\Rightarrow\ \boldsymbol{MN=\frac{a}{2}\sqrt{5+2\sqrt{2}}}$$

Now, the area of ΔMPN (see above fig-9), is given as follows

$$\frac{1}{2}(MN)(PQ)=\frac{1}{2}(PM)(PN)\sin(\pi-\tan^{-1}\sqrt{2})$$

$$\left(\frac{a}{2}\sqrt{5+2\sqrt{2}}\right)(PQ)=\left(\frac{a\sqrt{3}}{2}\right)\left(\frac{a}{\sqrt{2}}\right)\sin(\tan^{-1}\sqrt{2})$$

$$\sqrt{5+2\sqrt{2}}\,PQ=a\sqrt{\frac{3}{2}}\sin\left(\sin^{-1}\sqrt{\frac{2}{3}}\right)\quad\Rightarrow\quad \boldsymbol{PQ}=\frac{1}{\sqrt{5+2\sqrt{2}}}a\sqrt{\frac{3}{2}}\sqrt{\frac{2}{3}}=\boldsymbol{a\sqrt{\frac{5-2\sqrt{2}}{17}}}$$

Using Pythagorean theorem in right ΔPQM (see above fig-9), we get

$$\boldsymbol{MQ}=\sqrt{(PM)^2-(PQ)^2}=\sqrt{\left(\frac{a\sqrt{3}}{2}\right)^2-\left(a\sqrt{\frac{5-2\sqrt{2}}{17}}\right)^2}=\sqrt{\frac{3a^2}{4}-\frac{a^2(5-2\sqrt{2})}{17}}=\boldsymbol{\frac{a}{2}\sqrt{\frac{31+8\sqrt{2}}{17}}}$$

$$\Rightarrow\ \boldsymbol{QN}=MN-MQ=\frac{a}{2}\sqrt{5+2\sqrt{2}}-\frac{a}{2}\sqrt{\frac{31+8\sqrt{2}}{17}}=\frac{a}{2}\left(\sqrt{5+2\sqrt{2}}-\sqrt{\frac{31+8\sqrt{2}}{17}}\right)$$

$$=\frac{a}{2}\sqrt{\left(\sqrt{5+2\sqrt{2}}-\sqrt{\frac{31+8\sqrt{2}}{17}}\right)^2}=\frac{a}{2}\sqrt{\left(\frac{\sqrt{85+34\sqrt{2}}-\sqrt{31+8\sqrt{2}}}{\sqrt{17}}\right)^2}$$

$$= \frac{a}{2}\sqrt{\frac{\left(\sqrt{85+34\sqrt{2}}-\sqrt{31+8\sqrt{2}}\right)^2}{17}} = \frac{a}{2}\sqrt{\frac{85+34\sqrt{2}+31+8\sqrt{2}-2\sqrt{(85+34\sqrt{2})(31+8\sqrt{2})}}{17}}$$

$$= \frac{a}{2}\sqrt{\frac{116+42\sqrt{2}-2\sqrt{289(11+6\sqrt{2})}}{17}} = \frac{a}{2}\sqrt{\frac{116+42\sqrt{2}-2\sqrt{17^2(3+\sqrt{2})^2}}{17}}$$

$$= \frac{a}{2}\sqrt{\frac{116+42\sqrt{2}-34(3+\sqrt{2})}{17}} = \frac{a}{2}\sqrt{\frac{14+8\sqrt{2}}{17}} = \boldsymbol{a}\sqrt{\frac{\mathbf{7+4\sqrt{2}}}{\mathbf{34}}}$$

Using Pythagorean theorem in right ΔQMB (see above fig-8) , we get

$$\boldsymbol{BQ} = \sqrt{(MQ)^2+(MB)^2} = \sqrt{\left(\frac{a}{2}\sqrt{\frac{31+8\sqrt{2}}{17}}\right)^2+\left(\frac{a}{2}\right)^2} = \sqrt{\frac{a^2(31+8\sqrt{2}+17)}{68}} = \boldsymbol{a}\sqrt{\frac{\mathbf{12+2\sqrt{2}}}{\mathbf{17}}}$$

Using Pythagorean theorem in right ΔPQC (see fig-10) , we get

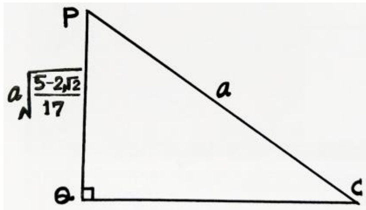

Figure 10: Right ΔPQC is obtained by dropping ⊥ to the plane of trapezium ABCD

$$\boldsymbol{QC} = \sqrt{(PC)^2-(PQ)^2} = \sqrt{(a)^2-\left(a\sqrt{\frac{5-2\sqrt{2}}{17}}\right)^2} = \sqrt{a^2-\frac{a^2(5-2\sqrt{2})}{17}} = \boldsymbol{a}\sqrt{\frac{\mathbf{12+2\sqrt{2}}}{\mathbf{17}}}$$

Using Pythagorean theorem in right ΔBEQ (see above fig-11) , we get

$$\boldsymbol{QE} = \sqrt{(BQ)^2-(BE)^2} = \sqrt{\left(a\sqrt{\frac{12+2\sqrt{2}}{17}}\right)^2-\left(\frac{a}{\sqrt{2}}\right)^2} = \sqrt{\frac{a^2(12+2\sqrt{2})}{17}-\frac{a^2}{2}}$$

$$= \sqrt{\frac{a^2(24+4\sqrt{2}-17)}{34}} = \sqrt{\frac{a^2(7+4\sqrt{2})}{34}} = \boldsymbol{a}\sqrt{\frac{\mathbf{7+4\sqrt{2}}}{\mathbf{34}}}$$

From computed values from fig-8 above, it is clear that $BQ = QC = QD = QA$ & $QE = QN$. Therefore ΔBQC, ΔCQD & ΔAQD are congruent isosceles triangles.

We know from **HCR's Theory of Polygon** that the **solid angle ($\boldsymbol{\omega}$), subtended by a right triangle OGH having perpendicular $\boldsymbol{p}$ & base $\boldsymbol{b}$ at any point P at a normal distance $\boldsymbol{h}$ on the vertical axis passing through the vertex O** (as shown in the fig-11)**,** is given by **HCR's Standard Formula-1** as follows

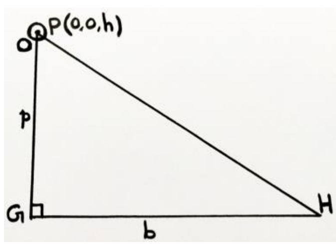

Figure-11: Point P lies at normal height $\boldsymbol{h}$ from vertex O of right $\Delta\boldsymbol{OGH}$ (⊥ to plane of paper).

$$\omega = \sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right) - \sin^{-1}\left(\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right)$$

Now, the solid angle $\omega_{\Delta QMB}$ subtended by right ΔQMB at the vertex P (see above fig-8) is obtained by substituting the corresponding values (as derived above) in above standard-1 formula i.e. base $b = MB = \frac{a}{2}$, perpendicular $p = MQ = \frac{a}{2}\sqrt{\frac{31+8\sqrt{2}}{17}}$ & normal height $h = PQ = a\sqrt{\frac{5-2\sqrt{2}}{17}}$ as follows

$$\omega_{\Delta QMB} = \sin^{-1}\left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{a}{2}\sqrt{\frac{31+8\sqrt{2}}{17}}\right)^2}}\right) - \sin^{-1}\left(\left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{a}{2}\sqrt{\frac{31+8\sqrt{2}}{17}}\right)^2}}\right)\left(\frac{a\sqrt{\frac{5-2\sqrt{2}}{17}}}{\sqrt{\left(a\sqrt{\frac{5-2\sqrt{2}}{17}}\right)^2+\left(\frac{a}{2}\sqrt{\frac{31+8\sqrt{2}}{17}}\right)^2}}\right)\right)$$

$$\omega_{\Delta QMB} = \sin^{-1}\left(\frac{\frac{a}{2}}{a\sqrt{\frac{12+2\sqrt{2}}{17}}}\right) - \sin^{-1}\left(\left(\frac{\frac{a}{2}}{a\sqrt{\frac{12+2\sqrt{2}}{17}}}\right)\left(\frac{a\sqrt{\frac{5-2\sqrt{2}}{17}}}{\frac{a\sqrt{3}}{2}}\right)\right)$$

$$\omega_{\Delta QMB} = \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{17}{12+2\sqrt{2}}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{3}}\sqrt{\frac{5-2\sqrt{2}}{12+2\sqrt{2}}}\right) = \mathbf{\sin^{-1}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) - \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{2-\sqrt{2}}{3}}\right)}$$

Similarly, the solid angle $\omega_{\Delta BEQ}$ subtended by right ΔBEQ at the vertex P (see above fig-8) is obtained by substituting the corresponding values (as derived above) in above standard formula-1 i.e. base $b = BE = \frac{a}{\sqrt{2}}$, perpendicular $p = QE = a\sqrt{\frac{7+4\sqrt{2}}{34}}$ & normal height $h = PQ = a\sqrt{\frac{5-2\sqrt{2}}{17}}$ as follows

$$\omega_{\Delta BEQ} = \sin^{-1}\left(\frac{\frac{a}{\sqrt{2}}}{\sqrt{\left(\frac{a}{\sqrt{2}}\right)^2+\left(a\sqrt{\frac{7+4\sqrt{2}}{34}}\right)^2}}\right) - \sin^{-1}\left(\left(\frac{\frac{a}{\sqrt{2}}}{\sqrt{\left(\frac{a}{\sqrt{2}}\right)^2+\left(a\sqrt{\frac{7+4\sqrt{2}}{34}}\right)^2}}\right)\left(\frac{a\sqrt{\frac{5-2\sqrt{2}}{17}}}{\sqrt{\left(a\sqrt{\frac{5-2\sqrt{2}}{17}}\right)^2+\left(a\sqrt{\frac{7+4\sqrt{2}}{34}}\right)^2}}\right)\right)$$

$$\omega_{\Delta BEQ} = \sin^{-1}\left(\frac{\frac{a}{\sqrt{2}}}{a\sqrt{\frac{12+2\sqrt{2}}{17}}}\right) - \sin^{-1}\left(\left(\frac{\frac{a}{\sqrt{2}}}{a\sqrt{\frac{12+2\sqrt{2}}{17}}}\right)\left(\frac{a\sqrt{\frac{5-2\sqrt{2}}{17}}}{\frac{a}{\sqrt{2}}}\right)\right)$$

$$\omega_{\Delta BEQ} = \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{17}{6+\sqrt{2}}}\right) - \sin^{-1}\left(\sqrt{\frac{5-2\sqrt{2}}{12+2\sqrt{2}}}\right) = \mathbf{\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{6-\sqrt{2}}{2}}\right) - \sin^{-1}\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)}$$

Thus, using symmetry in trapezium ABCD (see above fig-8), the solid angle ω_{ABCD} subtended by the trapezium ABCD at the vertex P of rhombicuboctahedron will be twice the solid angle ω_{MBCN} subtended by the trapezium MBCN at the vertex P, as follows

$$\omega_{ABCD} = 2\omega_{MBCN} = 2(\omega_{\Delta QMB}+\omega_{\Delta BEQ}+\omega_{\Delta CEQ}+\omega_{\Delta CNQ}) = 2(\omega_{\Delta QMB}+3\omega_{\Delta BEQ})$$

$$= 2\left\{\sin^{-1}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) - \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{2-\sqrt{2}}{3}}\right) + 3\left(\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{6-\sqrt{2}}{2}}\right) - \sin^{-1}\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)\right)\right\}$$

$$= 2\left\{\sin^{-1}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right) - \sin^{-1}\left(\frac{1}{2}\sqrt{\frac{2-\sqrt{2}}{3}}\right) + 3\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{6-\sqrt{2}}{2}}\right) - 3\sin^{-1}\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)\right\}$$

$$= 2\left\{\left(\sin^{-1}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)+\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{6-\sqrt{2}}{2}}\right)\right)-\left(\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{2-\sqrt{2}}{3}}\right)+\sin^{-1}\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)\right)\right.$$
$$\left.+2\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{6-\sqrt{2}}{2}}\right)-2\sin^{-1}\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)\right\}$$

Using formula: $\sin^{-1}x+\sin^{-1}y=\cos^{-1}\left(\sqrt{1-x^2}\sqrt{1-y^2}-xy\right)$ & $2\sin^{-1}x=\cos^{-1}(1-2x^2)\ \ \forall\ 0\le |x|,|y|\le 1$

$$= 2\left\{\cos^{-1}\left(\frac{\sqrt{10+\sqrt{2}}}{4}\cdot\frac{1}{2}\sqrt{\frac{2+\sqrt{2}}{2}}-\frac{\sqrt{6}-\sqrt{2}}{4}\cdot\frac{1}{2}\sqrt{\frac{6-\sqrt{2}}{2}}\right)-\cos^{-1}\left(\frac{1}{2}\sqrt{\frac{10+\sqrt{2}}{3}}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{1}{2}\sqrt{\frac{2-\sqrt{2}}{3}}\cdot\frac{\sqrt{2-\sqrt{2}}}{2}\right)\right.$$
$$\left.+\cos^{-1}\left(1-2\left(\frac{1}{2}\sqrt{\frac{6-\sqrt{2}}{2}}\right)^2\right)-\cos^{-1}\left(1-2\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)^2\right)\right\}$$

$$= 2\left\{\cos^{-1}\left(\frac{\sqrt{22+12\sqrt{2}}}{8\sqrt{2}}-\frac{6-\sqrt{2}}{8\sqrt{2}}\right)-\cos^{-1}\left(\frac{\sqrt{22+12\sqrt{2}}}{4\sqrt{3}}-\frac{2-\sqrt{2}}{4\sqrt{3}}\right)+\cos^{-1}\left(1-\frac{6-\sqrt{2}}{4}\right)-\cos^{-1}\left(1-\frac{2-\sqrt{2}}{2}\right)\right\}$$

$$= 2\left\{\cos^{-1}\left(\frac{\sqrt{(3\sqrt{2}+2)^2}-6+\sqrt{2}}{8\sqrt{2}}\right)-\cos^{-1}\left(\frac{\sqrt{(3\sqrt{2}+2)^2}-2+\sqrt{2}}{4\sqrt{3}}\right)+\cos^{-1}\left(\frac{-2+\sqrt{2}}{4}\right)-\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right\}$$

$$= 2\left\{\cos^{-1}\left(\frac{3\sqrt{2}+2-6+\sqrt{2}}{8\sqrt{2}}\right)-\cos^{-1}\left(\frac{3\sqrt{2}+2-2+\sqrt{2}}{4\sqrt{3}}\right)+\cos^{-1}\left(-\frac{2-\sqrt{2}}{4}\right)-\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right\}$$

$$= 2\left\{\cos^{-1}\left(\frac{4\sqrt{2}-4}{8\sqrt{2}}\right)-\cos^{-1}\left(\frac{4\sqrt{2}}{4\sqrt{3}}\right)+\boldsymbol{\pi}-\cos^{-1}\left(\frac{2-\sqrt{2}}{4}\right)-\frac{\pi}{4}\right\}$$

$$= 2\left\{\frac{3\pi}{4}+\cos^{-1}\left(\frac{2-\sqrt{2}}{4}\right)-\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)-\cos^{-1}\left(\frac{2-\sqrt{2}}{4}\right)\right\}$$

$$= 2\left\{\frac{3\pi}{4}-\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)\right\}$$

$$= \frac{3\pi}{2}-2\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)=\frac{3\pi}{2}-\cos^{-1}\left(2\left(\sqrt{\frac{2}{3}}\right)^2-1\right)=\frac{3\pi}{2}-\cos^{-1}\left(\frac{4}{3}-1\right)=\frac{3\pi}{2}-\cos^{-1}\left(\frac{1}{3}\right)$$

$$= \pi+\left(\frac{\pi}{2}-\cos^{-1}\left(\frac{1}{3}\right)\right)=\pi+\left(\sin^{-1}\left(\frac{1}{3}\right)\right)=\pi+\sin^{-1}\left(\frac{1}{3}\right)$$

It's worth noticing that the solid angle ω_V subtended by rhombicuboctahedron at its vertex P will be equal to the solid angle ω_{ABCD} subtended by the trapezium ABCD at the vertex P i.e. $\omega_V=\omega_{ABCD}=\pi+\sin^{-1}\left(\frac{1}{3}\right)$

Hence, the **solid angles $\boldsymbol{\omega_V}$ subtended by a rhombicuboctahedron at any of its 24 identical vertices** (at each of which one regular triangular & three square faces meet), is given as follows

$$\boldsymbol{\omega_V=\pi+\sin^{-1}\left(\frac{1}{3}\right)\ \mathrm{sr}\approx 3.481429563\ \mathrm{sr}} \qquad \ldots\ldots\ldots\ldots.(\mathbf{10})$$

Summary: Let there be a rhombicuboctahedron (small rhombicuboctahedron) having 8 congruent equilateral triangular faces & 18 congruent square faces, 48 edges each of length a & 24 identical vertices then all its important parameters are determined as tabulated below

Radius(R) of circumscribed sphere passing through all 24 vertices	$R = \frac{a}{2}\sqrt{5+2\sqrt{2}} \approx 1.398966326a$
Normal distances H_T & H_S of regular triangular & square faces from the centre of rhombicuboctahedron	$H_T = \frac{a(3+\sqrt{2})}{2\sqrt{3}} \approx 1.274273694\, a$ & $H_S = \frac{a(1+\sqrt{2})}{2} \approx 1.207106781\, a$
Surface area (A_s)	$A_s = 2a^2(9+\sqrt{3}) \approx 21.46410162\, a^2$
Volume (V)	$V = \frac{2a^3(6+5\sqrt{2})}{3} \approx 8.714045208\, a^3$
Mean radius (R_m) or radius of sphere having volume equal to that of the rhombicuboctahedron	$R_m = a\left(\frac{6+5\sqrt{2}}{2\pi}\right)^{1/3} \approx 1.276567352\, a$
Mid-radius (R_{md}) or radius of midsphere touching all 48 edges of the rhombicuboctahedron	$R_{md} = \sqrt{R^2 - \left(\frac{a}{2}\right)^2} = a\sqrt{\frac{2+\sqrt{2}}{2}} \approx 1.306562965\, a$
Dihedral angle θ_{TS} between adjacent equilateral triangular & square faces (i.e. having common side)	$\theta_{TS} = \pi - \tan^{-1}\frac{1}{\sqrt{2}} \approx 144.73^o$
Dihedral angle θ_{TV} between equilateral triangular & square faces with common vertex but no common side	$\theta_{TV} = \pi - \tan^{-1}\sqrt{2} \approx 125.26^o$
Dihedral angle θ_{SS} between any two adjacent square faces (i.e. having common side)	$\theta_{SS} = \frac{3\pi}{4} = 135^o$
Dihedral angle θ_{SV} between any two square faces meeting at the same vertex but no common side	$\theta_{SV} = 2\cos^{-1}\left(\sqrt{\frac{10-4\sqrt{2}}{17}}\cos 7.26^o\right) \approx 119.815^o$
Solid angles ω_T & ω_S subtended by equilateral triangular & square faces at the centre of rhombicuboctahedron	$\omega_T = 2\pi - 6\sin^{-1}\left(\frac{\sqrt{10+\sqrt{2}}}{4}\right)\text{ sr} \approx 0.24801961\text{ sr}$ $\omega_S = 4\sin^{-1}\left(\frac{2-\sqrt{2}}{4}\right)\text{ sr} \approx 0.587900762\text{ sr}$
Solid angle ω_V subtended by a rhombicuboctahedron at its vertex	$\omega_V = \pi + \sin^{-1}\left(\frac{1}{3}\right)\text{ sr} \approx 3.481429563\text{ sr}$

Note: Above articles had been *derived & illustrated* by **Mr H.C. Rajpoot** (**M Tech, Production Engineering**)

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