HCR's Rank Formula-2

Rank of Selective Linear permutations

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Introduction: We are much familiar with the linear permutations of digits & alphabetic letters but not much with the linear permutations of the non-algebraic articles having some distinguishable property like their shape, size, colour, surface-design etc. Although, it is very simple to replace each non-algebraic article in any linear permutation by a certain digit or alphabetic letter to find out its rank

Total no. of the linear permutations can easily be calculated using mathematical formula but their arrangement in the correct sequence (or in ranks) is difficult i.e. which one & where to place a given linear permutation in the set (group) according to its correct hierarchical rank. Like the numbers obtained by permuting certain digits, can be correctly arranged either in increasing order or decreasing order. Similarly words can be easily arranged in the correct alphabetic order or in reverse alphabetic order. HCR's Rank Formula-2 had been derived by the author Mr H.C. Rajpoot to find out the rank of any linear permutation randomly selected from a set of all the linear permutations obtained by permuting certain no. of the articles or elements while their repetition is allowed.

Selective Linear permutations: **These are the linear permutations which are obtained by permuting certain articles when their repetition is allowed.** Since these are obtained by the ways (or methods) of selecting the given articles with their replacement so called **selective linear permutations.** For example, total no. of 7-digit numbers formed by using 2, 5, 6 is equal to $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^7 = 2187$ if the repetition of the digits is allowed. We are here interested to study these selective linear permutations by finding out their ranks & arranging them according to the respective ranks.

Set of Linear Permutations: It is a group of all the possible linear permutations, each having equal no. of the articles, arranged in their hierarchical ranks (i.e. mathematically correct order of priority on the basis of pre-defined linear sequence of all the given articles).

Pre-defined Order: It is the linear sequence (arrangement) of certain articles in the correct order of priority.

A pre-defined order of given articles is always required for ranking of all the linear permutations of a set. All the linear permutations are arranged according to this pre-defined order of the given articles.

For ease of understanding, **order of priority** of articles is indicated by arrows from **left to right** on the basis of priority (any easily distiguishable property of articles) which may be increasing or decreasing or same in nature

Pre-defined linear arrangement of digits in increasing order

$\mathbf{0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9}$

Pre-defined linear arrangement of digits in decreasing order

$\mathbf{9 \to 8 \to 7 \to 6 \to 5 \to 4 \to 3 \to 2 \to 1 \to 0}$

Pre-defined linear arrangement of letters in correct alphabetic order

$\boldsymbol{A \to B \to C \to D \dots\dots\dots K \to L \to M \dots\dots\dots Q \to R \to S \dots\dots . W \to X \to Y \to Z}$

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<u>Pre-defined linear arrangement</u> of letters in <u>reverse alphabetic order</u>

$$\boldsymbol{Z \to Y \to X \to W \dots\dots\dots S \to R \to Q \dots\dots\dots M \to L \to K \dots\dots. D \to C \to B \to A}$$

Example: 1. Digits can be arranged either in increasing or in decreasing order. This order is called pre-defined order. Consider the digits 3, 9, 5, 3, 3, 1 which are arranged in their increasing order, keeping similar ones together, as follows

$$1 \to 3 \to 3 \to 3 \to 5 \to 9$$

2. Alphabetic letters are arranged in their alphabetic order or reverse order. This alphabetic order is the pre-defined order for the given alphabetic letters. Consider the letters L, A, H, P, P, N, D, D which are arranged in their alphabetic order, keeping similar ones together, as follows

$$A \to D \to D \to H \to L \to N \to P \to P$$

3. Non-algebraic things can be arranged in any order depending on our desire or preference of their property like shape, size, colour & value etc. Thus, we can select or make any pre-defined order of such non-algebraic articles. **Some linear arrangements of non-algebraic articles in their hierarchical ranks**

∟ → ® → Ⓣ → © → O → ⊥ → ∠ $(Having\ different\ shapes)$

© → © → © → © → © → © → © → © → © $(Having\ different\ colours)$

Ⓣ → $\boldsymbol{M}$ → $\boldsymbol{\Gamma}$ → $\boldsymbol{M}$ → ⊥ → ® → O → O $(Having\ different\ shapes\ \&\ colours)$

$\boldsymbol{Formerity\ (F)}$: $Formerity\ (F)$ of any article in a given linear permutation is equal to the no. of the distinct articles appearance before (left side) in the **pre-defined linear sequence of all the distinct articles**. **$\boldsymbol{Formerity\ (F)}$ of any article remains constant irrespective of its repetition (frequency) in a given linear permutation because it depends only on the pre-defined linear sequence of articles which is unique for whole the set of the linear permutations having equal no. of the articles.**

Permutation Value: Permutation Value (PV) of $\boldsymbol{i^{th}}$ article from the first (left most) place in a given linear permutation having total x no .of the articles, is given as

$$\boldsymbol{PV = F_i \times n^{x-i}}$$

Where,
$i = order\ of\ place\ of\ that\ article\ from\ the\ first\ (left\ most)\ place\ in\ the\ given\ linear\ permutation$

Example: Consider a 9-digit number having non-zero digits 787747443 it has four distinct digits 3, 4, 7 & 8

$$\therefore\ total\ no.of\ the\ distinct\ articles, n = 4$$

$$total\ no.of\ the\ articles\ in\ the\ permutation, x = total\ no.of\ the\ digits\ in\ 787747443 = 9$$

$$these\ distinct\ digits\ has\ following\ \boldsymbol{pre\ defined\ linear\ sequence}: 3 \to 4 \to 7^* \to 8$$

Now, consider any digit say 7 which lies at 6^{th} place from the left most place in 787747^*443 (as **labelled**) hence take $i = 6$ & there are two digits 3 & 4 appearing before it in pre-defined linear sequence (as **labelled**)

$$Hence, formerity\ of\ labelled\ 7,\ \ F(7^*) = 2$$

Now, the permutation value (PV) of labelled 7 can be given as

$$PV(7^*) = F(7^*) \times n^{x-i} = 2 \times 4^{9-6} = 2 \times 4^3 = 128$$

Conditions of Application of HCR's Formula-2:

1. All the articles are equally likely to occur at all the places in the linear permutations of a set.
2. Repetition of the articles is allowed i.e. any article can repeat or appear any times in any permutation.
3. All the linear permutations obtained are equally important, each of them having equal no. of the articles & assumed to be arranged in their correct ranks i.e. we can't remove any permutation from the set or all the linear permutations are assumed to be present in the set (group) of all the possible linear permutations obtained by certain articles.

HCR's Rank Formula-2: According to **HCR's Rank Hypothesis**, **if the repetition of n no. of the distinct articles is allowed then the rank of any linear permutation, given or randomly selected from a set (group) of all the linear permutations each having x no. of the articles (repeated or non-repeated or both) & arranged in the correct order, is given by the sum of unity (1) & Permutation Values (PV's) of all x no. of the articles in that linear permutation as follows**

$$R(linear\ permutation)$$

$$= 1 + \{F_1 \times n^{(x-1)} + F_2 \times n^{(x-2)} + F_3 \times n^{(x-3)} + \dots\dots\dots\dots\dots + F_{x-2} \times n^2 + F_{x-1} \times n^1 + F_n \times n^0\}$$

$$\Rightarrow \boldsymbol{R(linear\ permutation) = 1 + \sum_{i=1}^{i=x} F_i \times n^{(x-i)}}$$

Where,

$\boldsymbol{F_i = Formerity}\ of\ i^{th}\ article\ from\ the\ first\ (left\ most)\ place\ in\ the\ given\ linear\ permutation$

$\boldsymbol{= no.of\ the\ articles\ appearing\ before\ any\ article\ in\ pre\ defined\ linear\ sequence\ of\ all\ the\ articles}$

$n = total\ no. of\ the\ distinct\ articles\ taken\ for\ all\ the\ permutations\ in\ the\ set\ \ or$

$= maximum\ no. of\ the\ distinct\ articles\ appearing\ in\ all\ the\ permutations\ in\ the\ set$ &

$x = no. of\ the\ articles\ in\ the\ given\ (or\ randomly\ selected)\ linear\ permutation$

Let there be n no. of the distinct articles (i.e. all are equally significant & having distinguishable appearances). Now if the repetition of these articles is allowed then the total no. (N_t) of the linear permutations each having x no. of the articles is given as

$$N_t = no. of\ ways\ of\ filling\ x\ no. of\ the\ places\ by\ n\ no. of\ the\ articles\ while\ repetition\ is\ allowed$$

$$= n \times n \times n \times n \dots\dots\dots\dots\dots\dots\dots \times n \times n\ \ (x\ times) = n^x$$

$$\Rightarrow \boldsymbol{N_t = n^x}$$

Above result is applicable for equally significant articles in the linear permutations like for the numbers having only significant digits or non-zero digits.

Note: If there are total n no. of the distinct articles out of which n_o no. of the articles are non-significant articles (like zero-digit) which must not appear at the first (left most) place in any of the linear permutations of the set then the total no. $(\mathrm{N_t})$ of the significant linear permutations each having x no. of the articles

$$N_t = (total\ no. of\ the\ linear\ permutations) - (non\ significant\ linear\ permutations\ (N_o))$$

Let's calculate total no. (N_o) of the significant linear permutations each having x no. of the articles, assuming that the first place of each permutation is occupied by any of the non-significant articles, as follows

$$\Rightarrow N_o = n_o \times [no. of\ ways\ of\ filling (x-1)\ places\ by\ n\ no. of\ the\ articles\ (repetition\ allowed)]$$

$$= n_o \times [n \times n \times n \times n \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. \times n \times n \ \left((x-1)\ times\right)] = n_o n^{x-1}$$

$$(since,\ \ first\ (left\ most)\ place\ is\ already\ occupied\ by\ the\ non\ significant\ articles\ )$$

$$\therefore\ N_t = n^x - n_o n^{x-1} = (n - n_o) n^{x-1}$$

$$\Rightarrow \boldsymbol{N_t = (n - n_o) n^{x-1}} \qquad \forall\ \boldsymbol{n \geq n_o}$$

Above result is extremely useful for finding out the ranks of numbers having both zero & non-zero digits.

If we are to calculate the rank of any of the linear permutations in reverse order then we first calculate rank in the correct order by using HCR's Rank Formula-2 & then the rank in reverse order is given as

$$\boldsymbol{Rank\ in\ reverse\ order} = [\boldsymbol{Total\ no. of\ linear\ permutations}] - [\boldsymbol{Rank\ in\ correct\ order}] + \boldsymbol{1}$$

Verification of HCR's Rank Formula-2: Although, this formula has no mathematical derivation but it can be verified by the general result obtained by it. Let there be n no. of the distinct articles $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \ldots \ldots \ldots \ldots \Theta_{n-2}, \Theta_{n-1}$ & Θ_n these are equally significant at all the places in all the linear permutations of the set. Now if the repetition of the articles is allowed then the total no. (N_t) of the linear permutations, each having x no. of the articles, is given as

$$N_t = n^x$$

Now, let the **pre-defined linear sequence of all these distinct articles** be as follows

$$\Theta_1 \rightarrow \Theta_2 \rightarrow \Theta_3 \rightarrow \Theta_4 \rightarrow \ \ldots \ldots \ldots \ldots \rightarrow \Theta_{n-2} \rightarrow \Theta_{n-1} \rightarrow \Theta_n$$

It is clear from the above pre-defined sequence that all the linear permutation can be arranged in their ranks. Set of all these linear permutations, each having x no. of the articles, arranged in the correct ranks can be tabulated as follows

Selective Linear Permutation	Rank (order of priority)
$\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1 \ldots \ldots \ldots \Theta_1\Theta_1\Theta_1\Theta_1\ (\boldsymbol{x\ times})$	1
$\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1 \ldots \ldots \ldots \Theta_1\Theta_1\Theta_1\Theta_2\ (\boldsymbol{x\ times})$	2
$\Theta_1\Theta_1\Theta_1\Theta_1\Theta_1 \ldots \ldots \ldots \Theta_1\Theta_1\Theta_1\Theta_3\ (\boldsymbol{x\ times})$	3
………………………………………………………………………	………………………………
………………………………………………………………………	………………………………
$\Theta_n\Theta_n\Theta_n\Theta_n\Theta_n \ldots \ldots \ldots \Theta_n\Theta_n\Theta_n\Theta_{n-2}\ (\boldsymbol{x\ times})$	$n^x - 2$
$\Theta_n\Theta_n\Theta_n\Theta_n\Theta_n \ldots \ldots \ldots \Theta_n\Theta_n\Theta_n\Theta_{n-1}\ (\boldsymbol{x\ times})$	$n^x - 1$
$\Theta_n\Theta_n\Theta_n\Theta_n\Theta_n \ldots . \ldots \ldots \Theta_n\Theta_n\Theta_n\Theta_n\ (\boldsymbol{x\ times})$	n^x

It is clear from table, last linear permutation in the set is $\Theta_n\Theta_n\Theta_n\Theta_n\Theta_n \dots\dots\dots \Theta_n\Theta_n\Theta_n\Theta_n$ $(x\ times)$. Hence, its rank must be n^x($=$ $total\ no. of\ the\ linear\ permutations\ in\ the\ set$).

$Formerity\ of\ any\ article\ \Theta_n$ in the permutation $\Theta_n\Theta_n\Theta_n\Theta_n\Theta_n \dots\dots\dots \Theta_n\Theta_n\Theta_n\Theta_n\ (x\ times)$

$$F(\Theta_n) = no.\ of\ the\ articles\ appearing\ before\ \Theta_n\ \ in\ above\ pre\ defined\ sequence\ of\ all\ the\ articles$$

$$F(\Theta_n) = n-1 \quad (since,\ \ there\ are\ (n-1)\ articles\ appearing\ before\ \Theta_n\ in\ \boldsymbol{pre\ defined\ sequence})$$

Now using HCR's Formula-2 to calculate its rank by adding unity (1) & the permutation values of all the x no. of the articles in that linear permutation (taken from the above set or table) as follows

$$R(\Theta_n\Theta_n\Theta_n\Theta_n\Theta_n \dots\dots\dots \Theta_n\Theta_n\Theta_n\Theta_n\ (x\ times)) = 1 + \sum_{i=1}^{i=x} F_i \times n^{(x-i)}$$

$$= 1 + \{(n-1)\times n^{(x-1)} + (n-1)\times n^{(x-2)} + \dots\dots\dots\dots.. + (n-1)\times n^{(x-x+1)} + (n-1)\times n^{(x-x)}\}$$

$$= 1 + (n-1)\{n^{(x-1)} + n^{(x-2)} + n^{(x-3)}\ \dots\dots\dots\dots..n^2 + n + 1\}$$

$$= 1 + (n-1)\{sum\ of\ x\ terms\ of\ a\ G.P.\ having\ a = 1\ \&\ r = n\}$$

$$= 1 + (n-1)\left\{\frac{a(r^x-1)}{r-1}\right\} = 1 + (n-1)\left\{\frac{1(n^x-1)}{n-1}\right\} = 1 + (n^x - 1) = n^x$$

$$= N_t = total\ no. of\ the\ linear\ permutations\ each\ having\ x\ no. of\ the\ articles$$

We find that both the above results are equal hence *HCR's Rank Formula-2 is verified*.

How to find out the rank of any linear permutation when repetition of articles is allowed

Working Steps:

Step 1: Arrange all the distinct articles (elements) in their correct linear sequence. It is called **per-defined linear sequence** of the articles.

Step 2: Find out the value of $Formerity\ (F)$ & then the permutation value for each of the articles in the given or randomly selected linear permutation starting from first (left most) article up to last (right most) article in that permutation with the help of **per-defined linear sequence of articles (like increasing or decreasing order of digits, alphabetic order of letters etc. are pre-defined sequences well known to all)**.

Step 4: Now, use **Rank Formula** by adding unity (1) & the permutation values (PV's) of all the articles in that linear permutation to find out its rank.

Rank of Number having zero-digit: Rank of number having zero digit is calculated by using rank formula-2 first by assuming zero as the significant digit & then subtracting total non-significant numbers (having first place as zero) from the rank to get the correct rank of that number. Hence it is given as

$$R(Number\ with\ zero\ digit) \\ = (Rank\ assuming\ zero\ as\ significant\ digit) - (non\ significant\ numbers)$$

$$\boldsymbol{R(Number\ with\ zero\ digit) = 1 + \sum_{i=1}^{i=x} F_i \times n^{(x-i)} - n^{x-1}}$$

Illustrative Examples with detailed explanation

❖ Rank of Numbers

Example 1: Let's find out the rank in increasing (↑) & decreasing (↓) orders of a number 63328839 randomly selected from a set of all the 8-digit numbers obtained by permuting 1, 2, 3, 4, 6, 8 & 9 **while the repetition of digits is allowed**.

Sol. Here, $total\ no. of\ distinct\ digits\ used\ in\ all\ the\ linear\ permutations\ of\ the\ set,\ n = 7$ &

$No. of\ digits\ in\ the\ given\ number\ 63328839\ ,\ x = 8 \quad (since,\ all\ the\ numbers\ have\ 8\ digits).$

Now, let's follow the steps below

Step 1: Arrange all the distinct digits 1, 2, 3, 4, 6, 8 & 9 in **increasing order** or **pre-defined linear sequence** as follows

$$\mathbf{1 \to 2 \to 3 \to 4 \to 6 \to 8 \to 9}$$

Step 2: Now, find out value of $Formerity\ (F)$ & permutation value (PV) for each of eight digits in the given number 63328839 starting from the left most up to the right most digit by using the definitions of $Formerity\ (F)$ & $Permutation\ Value\ (PV)$ as follows

$\boldsymbol{Formerity\ of\ first\ (left\ most, i = 1)\ digit}\ 6\ (labelled)\ in\ \mathbf{6}3328839\ i.e.\ F(6)$

$= no. of\ the\ digits\ appearing\ before\ 6\ in\ above\ pre\ defined\ sequence\ (i.e.\ increasing\ order)$

$\Rightarrow F(6) = 4 \quad (since,\ there\ are\ four\ digits\ appearing\ before\ 6\ in\ above\ pre\ defined\ sequence)$

Now, the permutation value (PV) of labelled digit 6 is calculated as

$$\boldsymbol{PV}(\mathbf{6}) = F(6) \times n^{x-i} = 4 \times 7^{8-1} = 4 \times 7^7 = \mathbf{3294172}$$

$\boldsymbol{Formerity\ of\ next\ (i = 2)\ digit}\ 3\ (labelled)\ in\ 6\mathbf{3}328839\ i.e.\ F(3)$

$= no. of\ the\ digits\ appearing\ before\ 3\ in\ above\ pre\ defined\ sequence\ (i.e.\ increasing\ order)$

$\Rightarrow F(3) = 2 \quad (since,\ there\ are\ two\ digits\ appearing\ before\ 3\ in\ above\ pre\ defined\ sequence)$

Now, the permutation value (PV) of labelled digit 3 is calculated as

$$\boldsymbol{PV}(\mathbf{3}) = F(3) \times n^{x-i} = 2 \times 7^{8-2} = 2 \times 7^6 = \mathbf{235298}$$

$\boldsymbol{Formerity\ of\ next\ (i = 3)\ digit}\ 3\ (labelled)\ in\ 63\mathbf{3}28839\ i.e.\ F(3) = 2\ \ (it's\ constant\ for\ digit\ 3)$

Now, the permutation value (PV) of labelled digit 3 is calculated as

$$\boldsymbol{PV}(\mathbf{3}) = F(3) \times n^{x-i} = 2 \times 7^{8-3} = 2 \times 7^5 = \mathbf{33614}$$

$\boldsymbol{Formerity\ of\ next\ (i = 4)\ digit}\ 2\ (labelled)\ in\ 633\mathbf{2}8839\ i.e.\ F(2)$

$= no. of\ the\ digits\ appearing\ before\ 2\ in\ above\ pre\ defined\ sequence\ (i.e.\ increasing\ order)$

$F(2) = 0 \quad (since,\ there\ is\ no\ digit\ appearing\ before\ 2\ in\ above\ pre\ defined\ sequence)$

Now, the permutation value (PV) of labelled digit 2 is calculated as

$$\boldsymbol{PV}(\mathbf{2}) = F(2) \times n^{x-i} = 0 \times 7^{8-4} = \mathbf{0}$$

$\boldsymbol{Formerity\ of\ next}\ (\boldsymbol{i} = \mathbf{5})\ \boldsymbol{digit}\ 8\ (labelled)\ in\ 6332\mathbf{8}839\ i.e.\ F(8)$

$$= no.of\ the\ digits\ appearing\ before\ 8\ in\ above\ pre\ defined\ sequence\ (i.e.\ increasing\ order)$$

$\Rightarrow\ F(8) = 5 \qquad (since,\ \ there\ are\ five\ digits\ appearing\ before\ 8\ in\ above\ pre\ defined\ sequence)$

Now, the permutation value (PV) of labelled digit 8 is calculated as

$$\boldsymbol{PV}(\mathbf{8}) = F(8) \times n^{x-i} = 5 \times 7^{8-5} = 5 \times 7^3 = \mathbf{1715}$$

$\boldsymbol{Formerity\ of\ next}\ (\boldsymbol{i} = \mathbf{6})\ \boldsymbol{digit}\ 8\ (labelled)\ in\ 63328\mathbf{8}39\ i.e.\ F(8) = 5\ \ (it's\ constant\ for\ digit\ 3)$

Now, the permutation value (PV) of labelled digit 8 is calculated as

$$\boldsymbol{PV}(\mathbf{8}) = F(8) \times n^{x-i} = 5 \times 7^{8-6} = 5 \times 7^2 = \mathbf{245}$$

$\boldsymbol{Formerity\ of\ next}\ (\boldsymbol{i} = \mathbf{7})\ \boldsymbol{digit}\ 3\ (labelled)\ in\ 633288\mathbf{3}9\ i.e.\ F(3) = 2\ \ (it's\ constant\ for\ digit\ 3)$

Now, the permutation value (PV) of labelled digit 3 is calculated as

$$\boldsymbol{PV}(\mathbf{3}) = F(3) \times n^{x-i} = 2 \times 7^{8-7} = 2 \times 7 = \mathbf{14}$$

$\boldsymbol{Formerity\ of\ next}\ (\boldsymbol{last}, \boldsymbol{i} = \mathbf{8})\ \boldsymbol{digit}\ 9\ (labelled)\ in\ 6332883\mathbf{9}\ i.e.\ F(9)$

$$= no.of\ the\ digits\ appearing\ before\ 9\ in\ above\ pre\ defined\ sequence\ (i.e.\ increasing\ order)$$

$\Rightarrow\ F(9) = 6 \qquad (since,\ \ there\ are\ six\ digits\ appearing\ before\ 9\ in\ above\ pre\ defined\ sequence)$

Now, the permutation value (PV) of labelled digit 9 is calculated as

$$\boldsymbol{PV}(\mathbf{9}) = F(9) \times n^{x-i} = 5 \times 7^{8-8} = 6 \times 1 = \mathbf{6}$$

Step 3: Now, the rank of given number 63328839 is calculated by using the HCR's Rank Hypothesis as follows

$$R(63328839) = 1 + (sum\ of\ PV's\ of\ all\ the\ digits\ of\ 63328839)$$

$$= 1 + \big(PV(6) + PV(3) + PV(3) + PV(2) + PV(8) + PV(8) + PV(3) + PV(9)\big)$$

$$= 1 + (3294172 + 235298 + 33614 + 0 + 1715 + 245 + 14 + 6) = \mathbf{3565065}$$

Above steps have been shown only for ease of understanding of the procedure followed to apply HCR's Formula-2. Now, we are able to directly apply this formula without showing the steps as shown above only by using the **pre-defined linear sequence of given articles**.

Note: Same procedure is followed in case any number (with zero & non-zero digits), alphabetic words & all other permutations of different objects having different shape, size, colour & surface design etc.

Direct Application of HCR's Rank Formula-2: Rank of given number 63328839 is directly calculated by using the HCR's Rank Formula-2 as follows

Arrange all the distinct digits 1, 2, 3, 4, 6, 8 & 9 in **increasing order** or **pre-defined linear sequence** as follows

$$\mathbf{1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 6 \rightarrow 8 \rightarrow 9}$$

$$R(63328839) = 1 + \sum_{i=1}^{i=x} F_i \times n^{x-i} = 1 + \sum_{i=1}^{i=8} F_i \times 7^{8-i}$$

$$= 1 + \left\{F_1 \times 7^{(8-1)} + F_2 \times 7^{(8-2)} + F_3 \times 7^{(8-3)} + F_4 \times 7^{(8-4)} + F_5 \times 7^{(8-5)} + F_6 \times 7^{(8-6)} + F_7 \times 7^{(8-7)} + F_8 \times 7^{(8-8)}\right\}$$

Now, set the values of $Formerity\ (F)\ for\ each\ of\ the\ digits\ using\ above\ pre\ defined\ linear\ sequence$

$$R(63328839) = 1 + \{F(6) \times 7^7 + F(3) \times 7^6 + F(3) \times 7^5 + F(2) \times 7^4 + F(8) \times 7^3 + F(8) \times 7^2 + F(3) \times 7^1 + F(9) \times 7^0\}$$

$$= 1 + \{4 \times 7^7 + 2 \times 7^6 + 2 \times 7^5 + 0 \times 7^4 + 5 \times 7^3 + 5 \times 7^2 + 2 \times 7^1 + 6 \times 7^0\}$$

$$\boldsymbol{R}(\mathbf{63328839}) = 1 + (3294172 + 235298 + 33614 + 0 + 1715 + 245 + 14 + 6) = \mathbf{3565065}$$

It means that the number 63328839, randomly selected from the set of all $N_t = 7^8 = 5764801$ eight-digit numbers, is lying at **3565065th** place in the increasing order (↑) as shown in the table below.

$upward\ arrow\ (\uparrow) \Rightarrow\ shows\ the\ increasing\ order\ of\ rank$

$downward\ arrow\ (\downarrow) \Rightarrow\ shows\ the\ decreasing\ order\ of\ rank$

While the rank of 63328839 in the decreasing order (↓) is given as

$$\boldsymbol{R}(\mathbf{63328839}\downarrow) = N_t - R(63328839\uparrow) + 1 = 5764801 - 3565065 + 1 = \mathbf{2199737}$$

All the 5764801 eight-digit numbers formed by using the digits 1, 2, 3, 4, 6, 8 & 9 with the repetition can be correctly arranged in increasing & decreasing orders according to the ranks as tabulated below

Selective Linear Permutation (8-digit number)	**Rank in increasing order (↑)**	**Rank in decreasing order (↓)**
11111111	1	5764801
11111112	2	5764800
11111113	3	5764799
11111114	4	5764798
11111116	5	5764797
……………… ………………	……………… ………………	……………… ………………
28991648	1527259	4237543
28991649	1527260	4237542
28991661	1527261	4237541
……………… ………………	……………… ………………	……………… ………………
34188994	2014093	3750709
34188996	2014094	3750708
34188998	2014095	3750707
……………… ………………	……………… ………………	……………… ………………
63328839	**3565065**	**2199737**
63328841	3565066	2199736
63328842	3565067	2199735
……………… ………………	……………… ………………	……………… ………………
86611238	4655608	1109194

86611239	4655609	1109193
86611241	4655610	1109192
……………… ………………	…………….. ……………..	……………… ………………
99999993	5764797	5
99999994	5764798	4
99999996	5764799	3
99999998	5764800	2
99999999	5764801	1

Example 2: Let's find out the rank in increasing (↑) & decreasing (↓) orders of a number 7059077009 randomly selected from a set of all the 10-digit numbers obtained by permuting 0, 5, 7 & 9 while the repetition of digits is allowed.

$$here, \quad n = total\ no.of\ the\ \textbf{distinct digits}\ for\ all\ the\ numbers\ (permutations) = 4$$

$$x = total\ no.of\ the\ digits\ in\ the\ given\ number\ 7059077009 = 10$$

Here, zero is non-significant digit hence the total no. of significant 10-digit numbers obtained by permuting the distinct digits 0, 5, 7 & 9 with the repetition is given as

$$N_t = (n - n_o)n^{x-1} = (4-1)4^{10-1} = 3 \times 4^9 = 786432$$

While, the total non-significant 10-digit numbers (i.e. zero digit occurs at first (left most) place) obtained by the distinct digits 0, 5, 7 & 9 with the repetition is given as

$$N_o = n^{x-1} = 4^9 = 262144$$

Now, arrange all the distinct digits 0, in **increasing order** or **pre-defined linear sequence** as follows

$$\mathbf{0 \rightarrow 5 \rightarrow 7 \rightarrow 9}$$

Now, assuming zero digit as the significant & finding out the permutation value for each of the digits with the help of above pre-defined sequence. Rank of 7059077009 in increasing order (↑) using rank formula-2 is given as

$$\Rightarrow R(7059077009) = 1 + \sum_{i=1}^{i=x} F_i \times n^{x-i} - n^{x-1} = 1 + \sum_{i=1}^{i=10} F_i \times 4^{10-i} - 4^{10-1}$$

$$= 1 + \{F_1 \times 4^{(10-1)} + F_2 \times 4^{(10-2)} + F_3 \times 4^{(10-3)} + F_4 \times 4^{(10-4)} + F_5 \times 4^{(10-5)} + F_6 \times 4^{(10-6)} + F_7 \times 4^{(10-7)} + F_8 \times 4^{(10-8)} + F_9 \times 4^{(10-9)} + F_{10} \times 4^{(10-10)}\} - 262144$$

Now, set the values of $Formerity\ (F)\ for\ each\ of\ the\ digits\ using\ above\ pre\ defined\ linear\ sequence$

$$R(7059077009) = \{F(7) \times 4^9 + F(0) \times 4^8 + F(5) \times 4^7 + F(9) \times 4^6 + F(0) \times 4^5 + F(7) \times 4^4 + F(7) \times 4^3 + F(0) \times 4^2 + F(0) \times 4^1 + F(9) \times 4^0\} - 262143$$

$$= \{2 \times 4^9 + 0 \times 4^8 + 1 \times 4^7 + 3 \times 4^6 + 0 \times 4^5 + 2 \times 4^4 + 2 \times 4^3 + 0 \times 4^2 + 0 \times 4^1 + 3 \times 4^0\} - 262143$$

$$\boldsymbol{R(7059077009)} = (524288 + 0 + 16384 + 12288 + 0 + 512 + 128 + 0 + 0 + 3) - 262143$$

$$= \mathbf{291460}$$

It means that the number 7059077009, randomly selected from the set of all $N_t = 3 \times 4^9 = 786432$ ten-digit numbers, is lying at 291460th place in the increasing order (↑) as shown in the table below.

While the rank of 7059077009 in the decreasing order (↓) is given as

$$\boldsymbol{R}(\mathbf{7059077009} \downarrow) = N_t - R(7059077009 \uparrow) + 1 = 786432 - 291460 + 1 = \mathbf{494973}$$

All 786432 ten-digit numbers formed by using the digits 0, 5, 7 & 9 with the repetition can be correctly arranged in increasing & decreasing orders according to the ranks as tabulated below

Selective Linear Permutation (10-digit number)	**Rank in increasing order** (↑)	**Rank in decreasing order** (↓)
5000000000	1	786432
5000000005	2	786431
5000000007	3	786430
5000000009	4	786429
5000000050	5	786428
………………… …………………..	……………… ………………	……………. …………….
5999055007	258371	528062
5999055009	258372	528061
5999055050	258373	528060
………………… …………………..	…………… ……………	……………. …………….
7059077009	**291460**	**494973**
7059077050	291461	494972
7059077055	291462	494971
………………….. …………………..	…………… ……………	……………. ……………
7990955905	511346	275087
7990955907	511347	275086
7990955909	511348	275085
………………… …………………	……………. …………….	……………… ………………
9077990099	569104	217329
9077990500	569105	217828
9077990505	569106	217327
………………….. …………………..	……………. …………….	………………. ……………….
9999999979	786428	5
9999999990	786429	4
9999999995	786430	3
9999999997	786431	2
9999999999	786432	1

Thus, it is very easy to apply rank formula-2 to calculate the rank of any number having zero or non-zero or both the digits.

The same procedure (as mentioned in the example-1) is used in case of any other articles like letters & all other objects with different appearances (as mentioned below)

❖ Rank of Words

Example 3: Let's find out the alphabetic rank of a word ACADEMIA randomly selected from a set of all the words, each having eight letters, obtained by permuting A, C, D, E, I & M **while the repetition (replacement) of letters is allowed**.

Sol. Here, $total\ no.of\ distinct\ letters\ used\ in\ all\ the\ linear\ permutations\ of\ the\ set,\ n = 6$ &

$No.of\ letters\ in\ the\ given\ word\ ACADEMIA,\ x = 8 \quad (since,\ all\ the\ words\ have\ 8\ letters).$

Now, let's follow the steps below

Step 1: Arrange all the distinct letters in A, C, D, E, I & M **alphabetic order** or **pre-defined linear sequence** as follows

$$\boldsymbol{A \rightarrow C \rightarrow D \rightarrow E \rightarrow I \rightarrow M}$$

Step 2: Now, find out value of $Formerity\ (F)$ & permutation value (PV) for each of eight letters in the given ACADEMIA starting from the left most up to the right most letter by using the definitions of $Formerity\ (F)$ & $Permutation\ Value\ (PV)$ as follows

$\boldsymbol{Formerity\ of\ first\ (left\ most, i = 1)\ letter}\ A\ (labelled) in\ \boldsymbol{A}CADEMIA\ \ i.e.\ F(A)$

$$= no.of\ the\ letters\ appearing\ before\ A\ in\ above\ pre\ defined\ alphabetic\ order$$

$$\Rightarrow F(A) = 0 \qquad (since,\ there\ is\ no\ letter\ appearing\ before\ A\ in\ above\ pre\ defined\ sequence)$$

Now, the permutation value (PV) of labelled letter A is calculated as

$$\boldsymbol{PV(A)} = F(A) \times n^{x-i} = 0 \times 6^{8-1} = \boldsymbol{0}$$

$\boldsymbol{Formerity\ of\ next\ (i = 2)\ letter}\ C\ (labelled)\ in\ A\boldsymbol{C}ADEMIA\ \ i.e.\ F(C)$

$$= no.of\ the\ letters\ appearing\ before\ C\ in\ above\ pre\ defined\ alphabetic\ order$$

$$\Rightarrow F(C) = 1 \qquad (since,\ there\ is\ one\ letter\ appearing\ before\ C\ in\ above\ pre\ defined\ sequence)$$

Now, the permutation value (PV) of labelled letter C is calculated as

$$\boldsymbol{PV(C)} = F(C) \times n^{x-i} = 1 \times 6^{8-2} = 6^6 = \boldsymbol{46656}$$

$\boldsymbol{Formerity\ of\ next\ (i = 3)\ letter}\ A\ (labelled)\ in\ AC\boldsymbol{A}DEMIA\ \ i.e.\ F(A) = 0\ \ (it's\ constant\ for\ A)$

Now, the permutation value (PV) of labelled letter A is calculated as

$$\boldsymbol{PV(A)} = F(A) \times n^{x-i} = 0 \times 6^{8-3} = \boldsymbol{0}$$

$\boldsymbol{Formerity\ of\ next\ (i = 4)\ letter}\ D\ (labelled)\ in\ ACA\boldsymbol{D}EMIA\ \ i.e.\ F(D)$

$$= no.of\ the\ letters\ appearing\ before\ D\ in\ above\ pre\ defined\ alphabetic\ order$$

$$\Rightarrow F(D) = 2 \qquad (since,\ there\ are\ two\ letters\ appearing\ before\ D\ in\ above\ pre\ defined\ sequence)$$

Now, the permutation value (PV) of labelled letter D is calculated as

$$\boldsymbol{PV(D)} = F(D) \times n^{x-i} = 2 \times 6^{8-4} = 2 \times 6^4 = \boldsymbol{2592}$$

$\boldsymbol{Formerity\ of\ next}\ (\boldsymbol{i}=\mathbf{5})\ \boldsymbol{letter}\ E\ (labelled)\ in\ ACAD\boldsymbol{E}MIA\ \ i.e.\ F(E)$

$$= no.of\ the\ letters\ appearing\ before\ E\ in\ above\ pre\ defined\ alphabetic\ order$$

$\Rightarrow F(E) = 3 \quad (since,\ \ there\ are\ three\ letters\ appearing\ before\ E\ in\ above\ pre\ defined\ sequence)$

Now, the permutation value (PV) of labelled letter E is calculated as

$$\boldsymbol{PV}(\boldsymbol{E}) = F(E) \times n^{x-i} = 3 \times 6^{8-5} = 3 \times 6^3 = \mathbf{648}$$

$\boldsymbol{Formerity\ of\ next}\ (\boldsymbol{i}=\mathbf{6})\ \boldsymbol{letter}\ M\ (labelled)\ in\ ACADE\boldsymbol{M}IA\ \ i.e.\ F(M)$

$$= no.of\ the\ letters\ appearing\ before\ M\ in\ above\ pre\ defined\ alphabetic\ order$$

$\Rightarrow F(M) = 5 \quad (since,\ \ there\ are\ five\ letters\ appearing\ before\ M\ in\ above\ pre\ defined\ sequence)$

Now, the permutation value (PV) of labelled letter M is calculated as

$$\boldsymbol{PV}(\boldsymbol{M}) = F(M) \times n^{x-i} = 5 \times 6^{8-6} = 5 \times 6^2 = \mathbf{180}$$

$\boldsymbol{Formerity\ of\ next}\ (\boldsymbol{i}=\mathbf{7})\ \boldsymbol{letter}\ I\ (labelled)\ in\ ACADEM\boldsymbol{I}A\ \ i.e.\ F(I)$

$$= no.of\ the\ letters\ appearing\ before\ I\ in\ above\ pre\ defined\ alphabetic\ order$$

$\Rightarrow F(I) = 4 \quad (since,\ \ there\ are\ four\ letters\ appearing\ before\ I\ in\ above\ pre\ defined\ sequence)$

Now, the permutation value (PV) of labelled letter **'I'** is calculated as

$$\boldsymbol{PV}(\boldsymbol{I}) = F(I) \times n^{x-i} = 4 \times 6^{8-7} = 4 \times 6 = \mathbf{24}$$

$\boldsymbol{Formerity\ of\ next}\ (\boldsymbol{last}, \boldsymbol{i}=\mathbf{8})\ \boldsymbol{letter}\ A\ (labelled)\ in\ ACADEMI\boldsymbol{A}\ \ i.e.\ F(A) =$
$0\ \ (it's\ constant\ for\ A)$

Now, the permutation value (PV) of labelled letter A is calculated as

$$\boldsymbol{PV}(\boldsymbol{A}) = F(A) \times n^{x-i} = 0 \times 6^{8-8} = \mathbf{0}$$

Step 3: Now, the rank of given word ACADEMIA is calculated by using the HCR's Rank Hypothesis as follows

$$R(ACADEMIA) = 1 + (sum\ of\ PV's\ of\ all\ the\ letters\ of\ ACADEMIA)$$

$$= 1 + \big(PV(A) + PV(C) + PV(A) + PV(D) + PV(E) + PV(M) + PV(I) + PV(A)\big)$$

$$= 1 + (0 + 46656 + 0 + 2592 + 648 + 180 + 24 + 0) = \mathbf{50101}$$

Or

Rank of the given word ACADEMIA is directly calculated by using the HCR's Rank Formula-2 as follows

Arrange all the distinct letters A, C, D, E, I & M in **alphabetic order** or **pre-defined linear sequence** as follows

$$\boldsymbol{A \rightarrow C \rightarrow D \rightarrow E \rightarrow I \rightarrow M}$$

$$R(ACADEMIA) = 1 + \sum_{i=1}^{i=x} F_i \times n^{x-i} = 1 + \sum_{i=1}^{i=8} F_i \times 6^{8-i}$$

$$= 1 + \{F_1 \times 6^{(8-1)} + F_2 \times 6^{(8-2)} + F_3 \times 6^{(8-3)} + F_4 \times 6^{(8-4)} + F_5 \times 6^{(8-5)} + F_6 \times 6^{(8-6)} + F_7 \times 6^{(8-7)} + F_8 \times 6^{(8-8)}\}$$

Now, set the values of $Formerity\ (F)\ for\ each\ of\ the\ letters\ using\ above\ pre\ defined\ linear\ sequence$

$$R(ACADEMIA) = 1 + \{F(A) \times 6^7 + F(C) \times 6^6 + F(A) \times 6^5 + F(D) \times 6^4 + F(E) \times 6^3 + F(M) \times 6^2 + F(I) \times 6^1 + F(A) \times 6^0\}$$

$$= 1 + \{0 \times 6^7 + 1 \times 6^6 + 0 \times 6^5 + 2 \times 6^4 + 3 \times 6^3 + 5 \times 6^2 + 4 \times 6^1 + 0 \times 6^0\}$$

$$\boldsymbol{R}(\boldsymbol{ACADEMIA}) = 1 + \{0 + 46656 + 0 + 2592 + 648 + 180 + 24 + 0\} = \mathbf{50101}$$

It means that the word ACADEMIA, randomly selected from the set of all $N_t = 6^8 = 1679616$ words each having eight letters, is lying at $\mathbf{50101}^{st}$ place in the alphabetic order as shown in the table below.

Word (Selective Linear Permutation)	Alphabetic Rank
AAAAAAAA	1
AAAAAAAC	2
AAAAAAAD	3
AAAAAAAE	4
AAAAAAAI	5
………………… …………………	…………. ………….
ACADEMEM	50100
ACADEMIA	50101
ACADEMIC	50102
………………… …………………	…………. ………….
CMMEDIAC	556562
CMMEDIAD	556563
CMMEDIAE	556564
………………… …………………	…………. ………….
IMEMMMEA	1384111
IMEMMMEC	1384112
IMEMMMED	1384113
………………… …………………	…………. ………….
MMMMMMMC	1679612
MMMMMMMD	1679613
MMMMMMME	1679614
MMMMMMMI	1679615
MMMMMMMM	1679616

Example 4: Let's find out the alphabetic rank of a word **MATHEMATICS** randomly selected from a set of all the words, each having 11 letters, obtained by permuting A, C, E, H, I, M, S & T **while the repetition (replacement) of letters is allowed**.

$$here, \quad n = total\ no. of\ the\ distinct\ letters\ for\ all\ the\ words\ in\ the\ set = 8$$

$$x = total\ no. of\ the\ letters\ in\ the\ given\ word\ MATHEMATICS = 11$$

Hence the total no. of 11-letter words obtained by permuting the distinct letters with the repetition or replacement is given as

$$\Rightarrow N_t = 8^{11} = 8589934592$$

Now, arrange all the distinct letters A, C, E, H, I, M, S & T in **alphabetic order** (**pre-defined linear sequence)** as follows

$$\boldsymbol{A \rightarrow C \rightarrow E \rightarrow H \rightarrow I \rightarrow M \rightarrow S \rightarrow T}$$

$$R(MATHEMATICS) = 1 + \sum_{i=1}^{i=11} F_i \times n^{x-i} = 1 + \sum_{i=1}^{i=11} F_i \times 8^{11-i}$$

$$= 1 + \{F_1 \times 8^{(11-1)} + F_2 \times 8^{(11-2)} + F_3 \times 8^{(11-3)} + F_4 \times 8^{(11-4)} + F_5 \times 8^{(11-5)} + F_6 \times 8^{(11-6)} + F_7 \times 8^{(11-7)} + F_8 \times 8^{(11-8)} + F_9 \times 8^{(11-9)} + F_{10} \times 8^{(11-10)} + F_{11} \times 8^{(11-11)}\}$$

$$= 1 + \{F(M) \times 8^{10} + F(A) \times 8^9 + F(T) \times 8^8 + F(H) \times 8^7 + F(E) \times 8^6 + F(M) \times 8^5 + F(A) \times 8^4 + F(T) \times 8^3 + F(I) \times 8^2 + F(C) \times 8^1 + F(S) \times 8^0\}$$

Now, set the values of $Formerity\ (F)\ for\ each\ of\ the\ letters\ using\ above\ pre\ defined\ linear\ sequence$

$\Rightarrow R(MATHEMATICS)$

$$= 1 + \{5 \times 8^{10} + 0 \times 8^9 + 7 \times 8^8 + 3 \times 8^7 + 2 \times 8^6 + 5 \times 8^5 + 0 \times 8^4 + 7 \times 8^3 + 4 \times 8^2 + 1 \times 8^1 + 6 \times 8^0\}$$

$$= 1 + 5368709120 + 0 + 117440512 + 6291456 + 524288 + 163840 + 0 + 3584 + 256 + 8 + 6$$

$$= 5493133071$$

It means that the word MATHEMATICS, randomly selected from the set of all $N_t = 8^{11} = 8589934592$ words each having 11 letters, is lying at $\mathbf{5493133071}^{st}$ place in the alphabetic order as shown in the table below.

Word (Selective Linear Permutation)	Alphabetic Rank
AAAAAAAAAAA	1
AAAAAAAAAAC	2
AAAAAAAAAAE	3
AAAAAAAAAAH	4
AAAAAAAAAAI	5
……………………………… ………………………………	…………… ……………
ESCEMCATEAT	2975108744
ESCEMCATECA	2975108745
ESCEMCATECC	2975108746

…………………………… ……………………………	…………. ………….
HSCEMEMTITE	**4048903995**
HSCEMENTITH	**4048903996**
HSCEMEMTITI	**4048903997**
…………………………… ……………………………	…………. ………….
MATHEMATICM	**5493133070**
MATHEMATICS	**5493133071**
MATHEMATICT	**5493133072**
…………………………… ……………………………	…………. ………….
MTSEMCEIEST	**6414436704**
MTSEMCEIETA	**6414436705**
MTSEMCEIETC	**6414436706**
…………………………… ……………………………	…………. ………….
TTTTTTTTTTH	**8589934588**
TTTTTTTTTTI	**8589934589**
TTTTTTTTTTM	**8589934590**
TTTTTTTTTTS	**8589934591**
TTTTTTTTTTT	**8589934592**

❖ Rank of Selective Linear Permutations of Non-algebraic Articles

It is rather simple to arrange the numbers in increasing or decreasing order & the words in the alphabetic order but it comparatively difficult to arrange the linear permutations of **non-algebraic articles** (**all other things except digits & letters**) in the correct order. For the analysis of linear permutations of such articles, **we have to pre-define a linear sequence of the given articles** & replace each article by a digit or a alphabetic letter to get an equivalent number or word. Rank of this number or word is simply calculated by using HCR's Rank Formula-2 (used in case of repetition) & this rank will be the rank of given linear permutation. Thus, it becomes easy to calculate the rank of linear permutations of non-algebraic articles with the help of numbers or words. Digits (there are only 10 digits to replace maximum 10 distinct articles) are used for small no .of the articles in a given linear permutation but letters (there are 26 letters in English Alphabet to replace maximum 26 distinct articles) can be used for higher no. of the articles. Although, combinations of digits (i.e. 2-digit, 3-digit or multi-digit numbers) can be used to replace very large (greater than 26) no. of the distinct articles but, here we would calculate the ranks with the help of words by replacing each of the articles by an alphabetic letter according to the pre-defined linear sequence.

Working Steps:

Step 1: Arrange all the given articles (all must be distinct because repetition is allowed) in a **pre-defined linear sequence (it depends on our desire, usefulness, aesthetic quality of articles)** according to the basis of priority i.e. any easily distinguishable physical property like shape, size, colour etc. among the given articles.

Step 2: Replace each of the given distinct articles by a letter in the same linear sequence as that of the articles i.e. fist (left most) article by **A**, next (second) by **B**, next (third) by **C** & so on depending upon the no. of articles.

Thus we get an **equivalent linear sequence of alphabetic letters & an equivalent word corresponding to each of the linear permutations of the set**.

Step 3: Calculate the rank of word equivalent to a given or selected linear permutation by using rank formula-2 with the help of equivalent linear sequence of alphabetic letters. This will be the rank of given linear permutation of non-algebraic articles. This procedure is applied to calculate rank of any linear permutation of non-algebraic things. It will be clear from the following examples.

Illustrative Examples

1. Articles having different colours

Example 5: Consider the following group of 5 distinct articles identical in shape & size but dissimilar in colour

ⓉⓉⓉⓉⓉ

Now, the total no. of the linear permutations, each having 9 articles, obtained by permuting the above five distinct articles with the repetition (replacement) is given as

$$N_t = 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^9 = 1953125$$

So far, all above 1953125 linear permutations are random i.e. all these have no order of arrangement. Hence, in order to arrange all these random linear permutations we have to predefine a linear sequence of all the distinct articles according to some aesthetic quality. It is obvious that the above articles are identical in shape & size, but each of the articles above differs from other ones in colour. Thus "**colour**" among all the articles is the most suitable **distinguishable property (basis of priority)**. According to the basis of colour & utility (like usefulness, aesthetics, significance etc.), let the **pre-defined linear sequence** be as follows

Ⓣ → Ⓣ → Ⓣ → Ⓣ → Ⓣ

Hence, we can easily find out the **first linear permutation**, having 9 articles, of the set just by repeating the **first article** Ⓣ at all the 9 places as follows: ⓉⓉⓉⓉⓉⓉⓉⓉⓉ $\Rightarrow$ $Rank\ is\ \mathbf{1}$

Similarly, we can find out the **last linear permutation**, having 9 articles, of the set just by repeating the **last article** Ⓣ at all the 9 places as follows: ⓉⓉⓉⓉⓉⓉⓉⓉⓉ $\Rightarrow$ $Rank\ is\ \mathbf{1953125}$

While, all other linear permutations, each having 9 articles, will lie between above two linear permutations

Now, randomly consider any linear permutation from the set to calculate its rank. Let it be as follows

ⓉⓉⓉⓉⓉⓉⓉⓉⓉ

Replace all the articles of the pre-defined sequence by alphabetic letters A, B, C, D & E respectively, we get an equivalent linear sequence as follows

Ⓣ $\equiv A,$ Ⓣ $\equiv B,$ Ⓣ $\equiv C,$ Ⓣ $\equiv D$ & Ⓣ $\equiv E$

$\Rightarrow$ Ⓣ → Ⓣ → Ⓣ → Ⓣ → Ⓣ $\equiv A \to B \to C \to D \to E$

$\&\ given\ linear\ permutation,$ ⓉⓉⓉⓉⓉⓉⓉⓉⓉ $\equiv DEACBECAC$

$$Here,\ \ n = total\ no. of\ the\ distinct\ articles = 5$$

$$x = total\ no. of\ articles\ in\ any\ permutation\ of\ the\ set = 9$$

Now, using HCR's Rank Formula-2, the rank of equivalent linear permutation $DEACBECAC$ in the set of all the permutations, each having 9 letters (articles) is calculated as follows

$$R(DEACBECAC) = 1 + \sum_{i=1}^{i=9} F_i \times n^{x-i} = 1 + \sum_{i=1}^{i=9} F_i \times 5^{9-i}$$

$= 1 + \{F_1 \times 5^{(9-1)} + F_2 \times 5^{(9-2)} + F_3 \times 5^{(9-3)} + F_4 \times 5^{(9-4)} + F_5 \times 5^{(9-5)} + F_6 \times 5^{(9-6)} + F_7 \times 5^{(9-7)} + F_8 \times 5^{(9-8)} + F_9 \times 5^{(9-9)}\}$

$$= 1 + \{F(D) \times 5^8 + F(E) \times 5^7 + F(A) \times 5^6 + F(C) \times 5^5 + F(B) \times 5^4 + F(E) \times 5^3 + F(C) \times 5^2 + F(A) \times 5^1 + F(C) \times 5^0\}$$

Now, set the values of $Formerity\ (F)\ for\ each\ of\ the\ letters\ using\ above\ \ linear\ sequence\ of\ letters$

$\Rightarrow R(DEACBECAC)$

$$= 1 + \{3 \times 5^8 + 4 \times 5^7 + 0 \times 5^6 + 2 \times 5^5 + 1 \times 5^4 + 4 \times 5^3 + 2 \times 5^2 + 0 \times 5^1 + 2 \times 5^0\}$$

$$= 1 + 1171875 + 312500 + 0 + 6250 + 625 + 500 + 50 + 0 + 2 = 1491803$$

$= Rank\ of$ ⓉⓉⓉⓉⓉⓉⓉⓉⓉ

It means that the linear permutation ⓉⓉⓉⓉⓉⓉⓉⓉⓉ randomly selected from the set of all $N_t = 5^9 = 1953125$ linear permutations, each having 9 articles, is lying at $\mathbf{1491803^{rd}}$ place in the correct order of priority as shown in the table below.

Equivalent Alphabetic Word	***Linear Permutation of articles***	***Rank of priority***
AAAAAAAAA	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	1
AAAAAAAAB	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	2
AAAAAAAAC	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	3
AAAAAAAAD	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	4
AAAAAAAAE	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	5
………………… …………………	……………………………… ………………………………	…………….. ……………..
BEDCBECAB	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	757427
BEDCBECAC	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	757428
BEDCBECAD	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	757429
………………… …………………	………………………………… …………………………………	…………….. ……………..
CDCEBADAE	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	1060080
CDCEBADBA	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	1060081
CDCEBADBB	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	1060082
………………… …………………	……………………………… ………………………………	…………….. ……………..
DEACBECAB	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	1491802
DEACBECAC	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	**1491803**
DEACBECAD	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	1491804
…………………	……………………………	…………….

…………………	……………………………	……………..
EEEEEEEEA	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	1953121
EEEEEEEEB	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	1953122
EEEEEEEEC	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	1953123
EEEEEEEED	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	1953124
EEEEEEEEE	ⓉⓉⓉⓉⓉⓉⓉⓉⓉ	1953125

2. Articles having different shapes & colours

Example 6: Consider the following group of 6 distinct articles dissimilar in shape, size & colour

Ⓣ θ ⊥ ◺ £ ¥

Now, the total no. of the linear permutations, each having 8 articles, obtained by permuting the above six distinct articles with the repetition (replacement) is given as

$$N_t = 6\times 6\times 6\times 6\times 6\times 6\times 6\times 6 = 6^8 = 1679616$$

So far, all above 1679616 linear permutations are random i.e. all these have no order of arrangement. Hence, in order to arrange all these random linear permutations we have to predefine a linear sequence of all the distinct articles according to some aesthetic quality. The **distinguishable property (basis of priority)** may be taken as the **shape, size or colour** whichever is suitable to us. According to the basis of shape & utility (like usefulness, aesthetics, significance etc.), let the **pre-defined linear sequence** be as follows

◺ → ¥ → Ⓣ → £ → ⊥ → θ

Now, replace each of the articles of above linear sequence by an alphabetic letter in the same order thus, thus we get an equivalent linear sequence of letters to calculate rank of any permutation

◺ $\equiv A$, ¥ $\equiv B$, Ⓣ $\equiv C$, £ $\equiv D$, ⊥ $\equiv E$ & θ $\equiv F$

$\Rightarrow$ ◺ → ¥ → Ⓣ → £ → ⊥ → θ $\equiv A \to B \to C \to D \to E \to F$

Now, by using HCR's Rank Formula-2 & above alphabetic sequence, rank of any of 1679616 linear permutations of the set can easily be calculated & all these are arranged in the correct ranks as tabulated

Equivalent Alphabetic Word	***Linear Permutation of articles***	***Rank of priority***
AAAAAAAA	◺◺◺◺◺◺◺◺	1
AAAAAAAB	◺◺◺◺◺◺◺¥	2
AAAAAAAC	◺◺◺◺◺◺◺Ⓣ	3
AAAAAAAD	◺◺◺◺◺◺◺£	4
AAAAAAAE	◺◺◺◺◺◺◺⊥	5
………………… …………………	…………………………… ……………………………	…………….. ……………..
BDEFFCAD	¥£⊥θθⓉ◺£	458644
BDEFFCAE	¥£⊥θθⓉ◺⊥	458645
BDEFFCAF	¥£⊥θθⓉ◺θ	458646

………………… …………………	………………………………… …………………………………	…………… ……………
CBEEFADF	Ⓣ ¥ ⊥ ⊥ θ ◺ £ θ	643920
CBEEFAEA	Ⓣ ¥ ⊥ ⊥ θ ◺ ⊥ ◺	643921
CBEEFAEB	Ⓣ ¥ ⊥ ⊥ θ ◺ ⊥ ¥	643922
………………… …………………	……………………………… ………………………………	…………… ……………
DFABDCBC	£ θ ◺ ¥ £ Ⓣ ¥ Ⓣ	1075113
DFABDCBD	£ θ ◺ ¥ £ Ⓣ ¥ £	1075114
DFABDCBE	£ θ ◺ ¥ £ Ⓣ ¥ ⊥	1075115
………………… …………………	…………………………… ……………………………	…………… ……………
EDFDCBAC	⊥ £ θ £ Ⓣ ¥ ◺ Ⓣ	1302951
EDFDCBAD	⊥ £ θ £ Ⓣ ¥ ◺ £	1302952
EDFDCBAE	⊥ £ θ £ Ⓣ ¥ ◺ ⊥	1302953
……………… ………………	…………………………… ……………………………	…………… ……………
FFFFFFFB	θ θ θ θ θ θ θ ¥	1679612
FFFFFFFC	θ θ θ θ θ θ θ Ⓣ	1679613
FFFFFFFD	θ θ θ θ θ θ θ £	1679614
FFFFFFFE	θ θ θ θ θ θ θ ⊥	1679615
FFFFFFFF	θ θ θ θ θ θ θ θ	1679616

Thus, HCR's Rank formula can be applied to calculate the rank of any linear permutation when the repetition of the articles (like digits, letters & all other objects having different shape, size, colour & other aesthetic quality) is allowed.

Note: Above formula had been derived & *illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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