Mathematical Analysis of Circum-inscribed Polygons: Circum-inscribed (C-I) Trapezium

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ABSTRACT

The circumscribed and the inscribed polygons are well known and mathematically well defined in the context of 2D-Geometry. The term '**Circum-inscribed Polygon**' has been proposed by the author and used as a new definition of the polygon which satisfies the conditions of a circumscribed polygon and an inscribed polygon together. In other words, the circum-inscribed polygon is a polygon which has both the inscribed and circumscribed circles. The newly defined circum-inscribed polygon has each of its sides touching a circle and each of its vertices lying on another circle. The most common examples of circum-inscribed polygon are triangle, regular polygon, trapezium with each of its non-parallel sides equal to the Arithmetic Mean (AM) of its parallel sides (called circum-inscribed trapezium) and right kite. This paper describes the mathematical derivations of the analytic formula to find out the different parameters in terms of AM and GM of known sides such as radii of circumscribed & inscribed circles, unknown sides, interior angles, diagonals, angle between diagonals, ratio of intersecting diagonals, perimeter, and area of circum-inscribed trapezium and right kite. Like an inscribed polygon, a circum-inscribed polygon always has all of its vertices lying on infinite number of spherical surfaces. All the analytic formulae have been derived using simple trigonometry and 2-dimensional geometry which can be used to analyse the complex 2D and 3D geometric figures such as cyclic quadrilateral and trapezohedron, and other polyhedrons.

Keywords: Circum-inscribed Polygon, Circum-inscribed (C-I) Trapezium, AM-GM Property, C-I Right kite

1. Introduction

In general, each side of a circumscribed polygon touches a circle. Similarly, each vertex of an inscribed polygon lies on another circle. There are certain polygons which satisfy both these definitions i.e. polygons which are both the circumscribed and inscribed polygon. The polygons which are both the circumscribed and inscribed polygon are called circum-inscribed (C-I) polygons. The new term 'circum-inscribed' combines the geometric properties and simultaneously satisfies the conditions of becoming both the circumscribed and the inscribed polygon. The examples of C-I polygons are triangle, regular polygon, trapezium with each of its non-parallel side equal to the mean of its parallel sides and the right kite which has two opposite right angles such that the equal sides are adjacent. In this paper, the focus will be on the new terms and definitions of polygon, and the mathematical derivations of the analytic formula of C-I trapezium and right kite to find out the various parameters such as radii of circumscribed & inscribed circles, unknown sides, interior angles, diagonals, perimeter, and area in terms of arithmetic mean (AM) and geometric mean (GM) of the known sides.

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2. Circum-inscribed (C-I) polygon

It is the polygon which is both circumscribed and inscribed by two distinct (concentric or non-concentric) circles. In other words, a polygon which has both the inscribed and the circumscribed circles is called circum-inscribed polygon. The new term '**circum-inscribed**' can be abbreviated by **C-I**.

In the context of 2-dimensional geometry, a polygon is called **C-I polygon** if and only if it satisfies both the conditions below

1. **Condition-1: Circumscribed circle:** A polygon can be inscribed by a circle only if the perpendicular bisectors of all of its sides are concurrent. This condition implies that each vertex of the polygon lies on a circle i.e. circumscribed circle. The polygon is called inscribed polygon.
2. **Condition-2: Inscribed circle**: A polygon can circumscribe a circle only if the bisectors of all of its interior angles are concurrent. This condition ensures that each side of the polygon is tangent to a circle i.e. inscribed circle. The polygon is called circumscribed polygon.

Some common examples of such polygons are

1. Triangle
2. Regular polygon
3. CI Trapezium (having each of its non-parallel sides equal to AM of its parallel sides)
4. Right kite

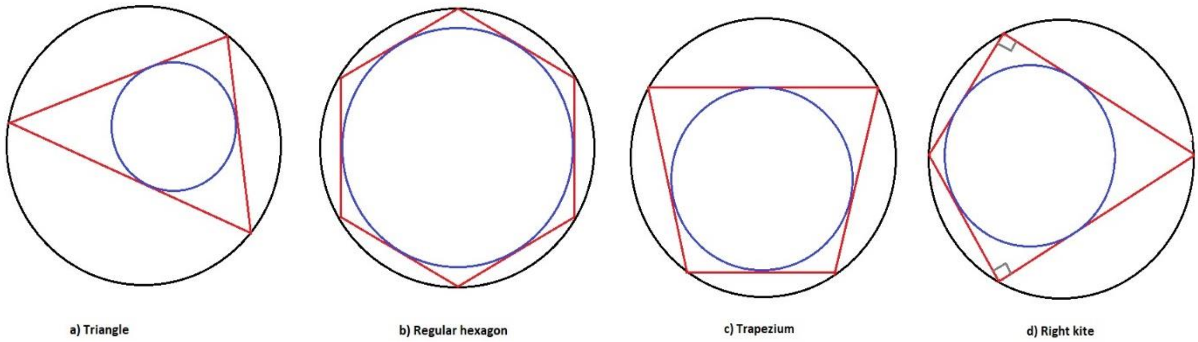

Figure-1: Triangle, regular hexagon (or polygon), trapezium and right kite are all circum-inscribed (C-I) polygons.

Thus, each triangle, regular polygon, C-I trapezium and right kite have both the inscribed and the circumscribed circles (as shown in the above figure-1). The analytic formula for determining the radii of the circumscribed and inscribed circles for the triangle are well known in terms of sides, and sides & angles. Therefore, the main focus will be on deriving the generalized and analytic formula for regular polygon, C-I trapezium and right kite.

2.1. Triangle

Let's consider a ΔABC having sides $a, b,$ and c (as shown in the figure-2 below). In order to ensure that all the vertices are equidistant from a point (internal or external), the perpendicular bisectors of all the sides AB, BC and AC are drawn which intersect each other at a certain point O i.e. called circumscribed centre which is equidistant from each of its vertices A, B, and C. Thus a circle with centre at the intersection point O and radius equal to OA, OB or OC always passes through each of the vertices. The circle passing through each of the vertices A, B, and C is called circumscribed circle of ΔABC (see the figure-2 below).

2.1.1. Radius of circumscribed circle

Let R be the radius of circumscribed circle of ΔABC. From the corollary that the angle subtended by any chord at the centre of circle is twice the angle subtended by it at any point on the corresponding segment.

$$\therefore\ \angle BOC = 2\angle BAC = 2A,\ \ \angle AOC = 2\angle ABC = 2B \ \text{ and } \angle AOB = 2\angle ACB = 2C$$

In right ΔOMB

$$\sin\angle BOM = \sin\frac{\angle BOC}{2} = \frac{BM}{OB}$$

$$\Rightarrow\ \sin A = \frac{a/2}{R}\ \Rightarrow\ R = \frac{a}{2\sin A}$$

$$\Rightarrow R = \frac{abc}{2bc\sin A} = \frac{abc}{4\left(\frac{1}{2}bc\sin A\right)} = \frac{abc}{4\Delta}$$

$$\left(\because\ \Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C\right)$$

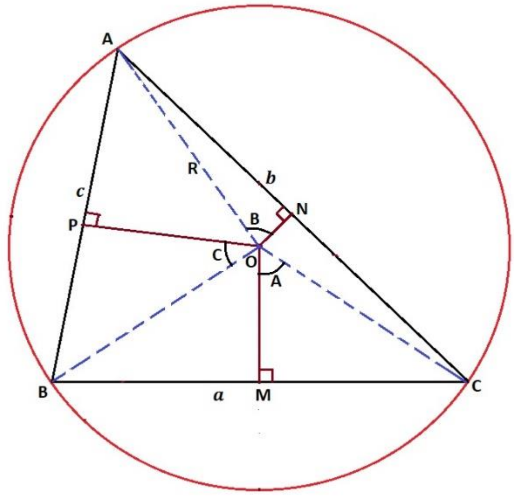

Figure-2: ΔABC is circumscribed by a circle with center O and radius R

Similarly, in right ΔONA

$$\sin\angle AON = \sin\frac{\angle AOC}{2} = \frac{AN}{OA}$$

$$\Rightarrow\ \sin B = \frac{b/2}{R}\ \Rightarrow\ R = \frac{b}{2\sin B}$$

Similarly, in right ΔOPA

$$\sin\angle AOP = \sin\frac{\angle AOB}{2} = \frac{AP}{OA}\ \Rightarrow\ \sin C = \frac{c/2}{R}\ \Rightarrow\ R = \frac{c}{2\sin C}$$

Therefore, the radius R of circumscribed circle of ΔABC with sides $a, b,$ and c is given by following formula

$$\boldsymbol{R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}}$$

2.1.2. Radius of inscribed circle

In order to ensure that all the sides are at an equal normal distance from an internal point, the bisectors of all the interior angles A, B, and C of ΔABC are drawn which intersect each other at a point I i.e. called inscribed centre which has equal normal distance from each of its sides AB, BC, and AC. Thus a circle with centre at the intersection point I and radius equal to OM, ON or OP always touches each of three sides. The circle touching each of the sides AB, BC, and AC is called inscribed circle of ΔABC (see the figure-3 below).

Let r be the radius of inscribed circle of ΔABC. Now, from Angle-Side-Angle congruence theorem, ΔIMB & ΔIPB, ΔIMC & ΔINC and ΔINA & ΔIPA are pair-wise congruent triangles. Let's assume that

$$MB = PB = x, \qquad MC = NC = y \quad \text{and} \quad \mathrm{NA} = \mathrm{PA} = \mathrm{z}$$

In right ΔABC

$$BM + MC = BC\ \Rightarrow\ x + y = a \qquad (i)$$

$$AN + NC = AC\ \Rightarrow\ y + z = b \qquad (ii)$$

$$AP + PB = AB\ \Rightarrow\ x + z = c \qquad (iii)$$

Now, adding the above Eq(i) and Eq(iii) and subtracting Eq(ii)

$$x + y + x + z - (y + z) = a + c - b$$

$$\Rightarrow \quad x = \frac{a + c - b}{2} = \frac{a + b + c - 2b}{2}$$
$$= \frac{a + b + c}{2} - b = s - b$$

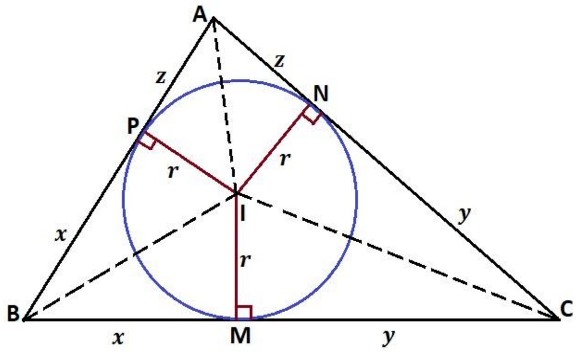

Figure-3: ΔABC inscribed a circle with center I and radius r

Where, s is semi-perimeter of ΔABC. Substituting the value of x into the above Eq(i) and Eq(iii)

$$\Rightarrow \quad y = a - x = a + b - s = s - c$$

$$\Rightarrow \quad z = c - x = c + b - s = s - a$$

$$\therefore \; MB = PB = x = s - b, \qquad MC = NC = y = s - c \quad \text{and} \quad \mathrm{NA} = \mathrm{PA} = \mathrm{z} = \mathrm{s} - \mathrm{a}$$

In right ΔIMB (see the above figure-3)

$$\tan \angle IBM = \tan \frac{B}{2} = \frac{IM}{MB} \qquad (\because \text{ BI is the bisector of } \angle ABC = B)$$

$$\Rightarrow \quad \tan \frac{B}{2} = \frac{r}{s - b} \quad \Rightarrow r = (s - b) \tan \frac{B}{2}$$

Similarly, in right ΔIMC (see the above figure-3)

$$\tan \angle ICM = \tan \frac{C}{2} = \frac{IM}{MC} \qquad (\because \text{ CI is the bisector of } \angle ACB = C)$$

$$\Rightarrow \quad \tan \frac{C}{2} = \frac{r}{s - c} \quad \Rightarrow r = (s - c) \tan \frac{C}{2}$$

Similarly, in right ΔINA (see the above figure-3)

$$\Rightarrow r = (s - a) \tan \frac{A}{2}$$

The area Δ of ΔABC can be determined by adding the areas of all pairwise-congruent triangles ΔIMB & ΔIPB, ΔIMC & ΔINC and ΔINA & ΔIPA as follows

$$\Delta = ([\Delta IMB] + [\Delta IPB]) + ([\Delta IMC] + [\Delta INC]) + ([\,\Delta INA] + [\,\Delta IPA])$$

$$\Delta = 2[\Delta IMB] + 2[\Delta IMC] + 2[\,\Delta INA] \qquad (\because \text{ Congruent triangles have equal area})$$

$$\Delta = 2\left(\frac{1}{2}(IM)(MB)\right) + 2\left(\frac{1}{2}(IM)(MC)\right) + 2\left(\frac{1}{2}(IN)(NA)\right)$$

$$\Delta = (r)(s - b) + (r)(s - c) + (r)(s - a) = r(3s - (a + b + c)) = r(3s - 2s)$$

$$\Delta = rs \quad \Rightarrow \quad r = \frac{\Delta}{s}$$

Therefore, the radius R of circumscribed circle of ΔABC with sides $a, b,$ and c is given by following formula

$$\boldsymbol{r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} = \frac{\Delta}{s}}$$

2.2. Regular polygon

It is the polygon which has all of its sides and interior angles equal. The bisectors of all the interior angles of regular polygon intersect one another at a single point (called in-centre). The perpendicular bisectors of all the sides of a regular polygon always intersect one another at a single point (called circum-centre). Thus a regular polygon always has both the inscribed and circumscribed circles (which are always concentric). Therefore, a regular polygon is always a circum-inscribed (C-I) polygon.

Now, consider a regular polygon $A_1A_2A_3 \dots . A_{n-1}A_n$ having n no. of sides each of length a. The perpendicular bisectors of all the sides are drawn which intersect one another at a point O i.e. circum-centre. Similarly, the bisectors of all the interior angles intersect one another at a single point O i.e. in-centre. In this case, the circum-centre and in-centre are both coincident. (as shown in the figure-4 below).

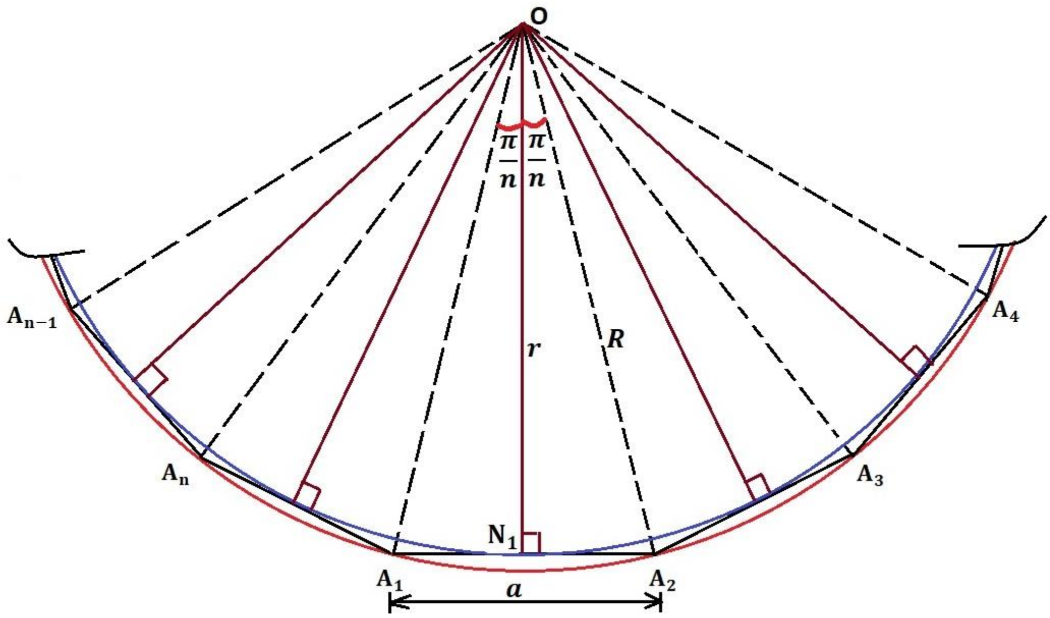

Figure-4: The perpendicular bisectors of all the sides and the bisectors of all the interior angles of a regular polygon $A_1A_2A_3 \dots . A_{n-1}A_n$ intersect one another at a single point O which is both circum-centre and in-centre.

Let R and r be the radii of circumscribed and inscribed circles respectively (as labelled with red and blue colours in the above figure-4).

In right ΔON_1A_2 (see the above figure-4)

$$\sin\angle N_1OA_2 = \sin\frac{\pi}{n} = \frac{N_1A_2}{OA_2} \qquad \left(\because ON_1 \text{ is the bisector of } \angle A_1OA_2 = \frac{2\pi}{n}\right)$$

$$\Rightarrow \quad \sin\frac{\pi}{n} = \frac{a/2}{R} \;\Rightarrow R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n} \qquad (\because N_1 \text{ is the midpoint of side } A_1A_2)$$

Similarly,

$$\tan\angle N_1OA_2 = \tan\frac{\pi}{n} = \frac{N_1A_2}{ON_1}$$

$$\Rightarrow \quad \tan\frac{\pi}{n} = \frac{a/2}{r} \;\Rightarrow r = \frac{a}{2}\cot\frac{\pi}{n}$$

Therefore, the radii R and r of circumscribed and inscribed circles of regular polygon having n no. of sides each of length a, are respectively as follows

$$\boldsymbol{R=\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}\ ,\ \ r=\frac{a}{2}\cot\frac{\pi}{n}\quad\Rightarrow\quad R=r\sec\frac{\pi}{n}}$$

The regular polygon with n no. of sides each of length a, is divided into n no. of congruent isosceles triangles when the centre O is joined to all the vertices of polygon (see the above figure-4). Thus, the area of regular polygon $A_1A_2A_3\ \dots.A_{n-1}A_n$ will be equal to the sum of areas of n no. of isosceles triangles congruent to isosceles ΔA_1OA_2. Therefore the area say Δ_n of regular polygon having n no. of sides each of length a is given as

$$\Delta_{\boldsymbol{n}}=n(\text{Area of }\Delta A_1OA_2)=n\left(\frac{1}{2}(A_1A_2)(ON_1)\right)=n\left(\frac{1}{2}(a)\left(\frac{a}{2}\cot\frac{\pi}{n}\right)\right)=\boldsymbol{\frac{1}{4}na^2\cot\frac{\pi}{n}}$$

2.3. Right kite

It is a quadrilateral having two opposite right angles and two pairs of equal adjacent sides. A right kite is formed by two congruent right triangles with a common hypotenuse. A right triangle can always be inscribed in a semicircle such that its hypotenuse coincides with the diameter of semicircle. Therefore, a right kite consisting of two congruent right triangles is always inscribed by a circle. It is noticeable in a right kite that the bisectors of the angles between equal sides both coincide with the longest diagonal and the bisectors of opposite right angles always intersect the longest diagonal at a single point. This implies that a right kite always has an inscribed circle.

Thus a right kite has both the inscribed and the circumscribed circles and hence it is always a circum-inscribed (C-I) polygon (as shown in the figure-5).

Now, consider a right kite ABCD having its unequal sides $AB=AD=a$ and $BC=CD=b$. The circum-centre O and the in-centre I can be found by drawing the perpendicular bisectors of all the sides and the bisectors of all interior angles. Join the circum-centre O and in-centre I to all the vertices and drop the perpendicular IM from in-centre I to the side BC (see the figure-5). Let R & r be the radii of circumscribed and inscribed circles respectively.

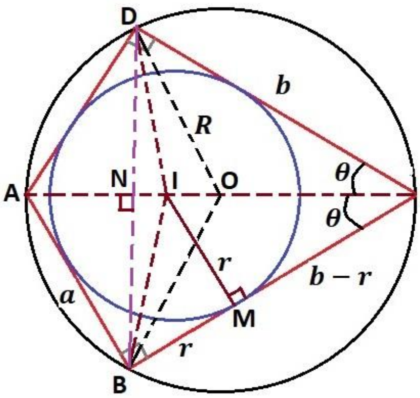

Figure-5: The perpendicular bisectors of all the sides and the bisectors of all the interior angles of (C-I) right kite ABCD intersect one another at the points O and I which are circum-centre and in-centre respectively.

The dotted straight line IB is the bisector of interior right $\angle ABC$.

$$\therefore\ \angle IBM=\frac{\angle ABC}{2}=\frac{\frac{\pi}{2}}{2}=\frac{\pi}{4}\ \Rightarrow\ \angle BIM=\frac{\pi}{2}-\angle IBM=\frac{\pi}{4}$$

$$\Rightarrow\ BM=IM=r\quad\Rightarrow\quad MC=BC-BM=b-r$$

In right ΔABC, using Pythagorean theorem,

$$AC=\sqrt{(AB)^2+(BC)^2}=\sqrt{a^2+b^2}$$

Now, from angle-angle similarity, the right ΔABC and ΔIMC are similar (see figure-5). Now, using property of the similar triangles ΔABC and ΔIMC as follows

$$\frac{AB}{BC}=\frac{IM}{MC}$$

$$\frac{a}{b}=\frac{r}{b-r}\quad\Rightarrow ab-ar=br\quad\Rightarrow r(a+b)=ab\quad\Rightarrow\ r=\frac{ab}{a+b}$$

Therefore, the radius r of inscribed circle of right kite having unequal sides of lengths a and b, is given as follows

$$\boldsymbol{r = \frac{ab}{a+b}}$$

The radius R of circumscribed circle of right kite having unequal sides of lengths a and b, is given as

$$\boldsymbol{R} = \frac{AC}{2} = \frac{\boldsymbol{\sqrt{a^2+b^2}}}{\boldsymbol{2}}$$

Now, using property of the similar right triangles ΔANB and ΔABC as follows

$$\frac{BN}{AB} = \frac{BC}{AC} \quad \Rightarrow \quad \frac{BN}{a} = \frac{b}{\sqrt{a^2+b^2}} \quad \Rightarrow \quad BN = \frac{ab}{\sqrt{a^2+b^2}}$$

$$\therefore \quad BD = BN + ND = BN + BN = 2BN = \frac{2ab}{\sqrt{a^2+b^2}}$$

Therefore, the diagonals of right kite having unequal sides of lengths a and b (see the above figure-5) is given as

$$\boldsymbol{AC = \sqrt{a^2+b^2}} \ , \ \boldsymbol{BD = \frac{2ab}{\sqrt{a^2+b^2}}}$$

The area Δ_{RK} of right kite ABCD will be equal to the sum of areas of congruent right triangles which is given as follows

$$\Delta_{\boldsymbol{RK}} = 2([\Delta ABC]) = 2\left(\frac{1}{2}ab\right) = \boldsymbol{ab} \qquad \text{([}\Delta\text{ABC] is area of }\Delta\text{ABC)}$$

The interior angles of right kite having unequal sides of lengths a and b (see the above figure-5) are given as

$$\boldsymbol{2\theta = 2\tan^{-1}\left(\frac{a}{b}\right)} \ , \ \boldsymbol{\pi - 2\theta = 2\tan^{-1}\left(\frac{b}{a}\right)}$$

2.4. Circum-inscribed (C-I) trapezium

It is the trapezium which circumscribes a circle and is inscribed by another circle. A C-I trapezium has both the inscribed and circumscribed circles. The perpendicular bisectors of all the sides of a C-I trapezium always intersect one another at a single point (called circum-centre). The bisectors of all the interior angles of a C-I trapezium intersect one another at a single point (called in-centre).

Now, consider a C-I trapezium ABCD having its parallel opposite sides AB and CD of length a and b respectively. The circum-centre O and the in-centre I can be found by drawing the perpendicular bisectors of all the sides and the bisectors of all interior angles. Join the circum-centre O to all the vertices and drop the perpendiculars from in-centre I to all the sides (see the figure-6 below). Let R & r be the radii of circumscribed and inscribed circles respectively.

Let c be the length of each of equal non-parallel sides AD and BC. Let $\angle BAD = 2\theta$ be the interior angle between the side AB $= a$ and side AD $= c$ which is bisected by straight line AI (as shown in the figure-6 below).

2.4.1. In-radius (r)

In quadrilateral ADMN, the sum of all interior angles is 2π .

$$\angle NAD + \angle ADM + \angle DMN + \angle ANM = 2\pi$$

$$2\theta + \angle ADM + \frac{\pi}{2} + \frac{\pi}{2} = 2\pi$$

$$\angle ADM = \pi - 2\theta$$

$$\angle ADI + \angle IDM = \pi - 2\theta$$

$$\angle ADI + \angle ADI = \pi - 2\theta \qquad (\because \ \angle IDM = \angle ADI)$$

$$2\angle ADI = \pi - 2\theta \ \Rightarrow \angle ADI = \frac{\pi}{2} - \theta$$

$$\Rightarrow \ \angle DIP = \frac{\pi}{2} - \angle ADI = \theta$$

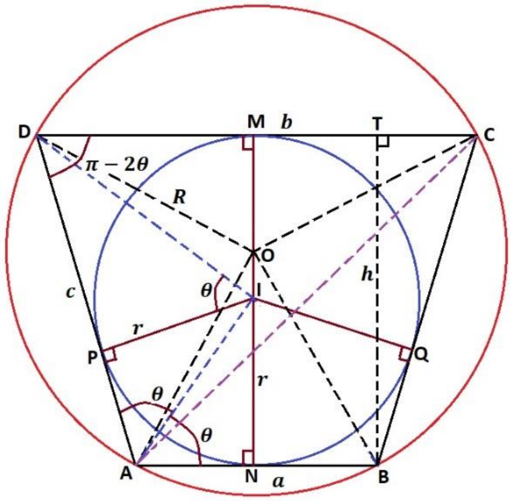

Figure-6: The perpendicular bisectors of all the sides and the bisectors of all the interior angles of C-I trapezium ABCD intersect one another at the points O and I which are circum-centre and in-centre respectively.

In right ΔIMD (see the figure-6)

$$\tan \angle DIM = \frac{DM}{IM} \quad \Rightarrow \quad \tan\theta = \frac{CD/2}{r} = \frac{b/2}{r} = \frac{b}{2r} \qquad \ldots \ldots \ldots \text{Eq}(1)$$

In right ΔINA

$$\tan \angle IAN = \frac{IN}{AN} \quad \Rightarrow \quad \tan\theta = \frac{r}{AB/2} = \frac{r}{a/2} = \frac{2r}{a} \qquad \ldots \ldots \ldots \text{Eq}(2)$$

Now, equating above Eq(1) and Eq(2)

$$\frac{b}{2r} = \frac{2r}{a} \ \Rightarrow 4r^2 = ab \quad \Rightarrow r = \frac{ab}{4} = \frac{\sqrt{ab}}{2}$$

Therefore, the radius r of inscribed circle of C-I trapezium having parallel sides of lengths a and b, is given as follows

$$\boldsymbol{r = \frac{\sqrt{ab}}{2} = \frac{GM}{2}}$$

Where, GM is geometric mean of parallel sides a and b.

2.4.2. Interior angles $(\boldsymbol{\theta, \pi - \theta})$

Substituting the value of r in the above Eq(1), the interior angle 2θ between non-parallel side c and parallel side a can be determined as follows

$$\tan\theta = \frac{b}{2r} = \frac{b}{2\cdot\frac{\sqrt{ab}}{2}} = \sqrt{\frac{b}{a}} \quad \Rightarrow \quad 2\theta = 2\tan^{-1}\left(\sqrt{\frac{b}{a}}\right)$$

The other interior angle $\angle ADC$ between non-parallel side c and parallel side b is given as follows

$$\angle ADC = \pi - 2\theta = \pi - 2\tan^{-1}\left(\sqrt{\frac{b}{a}}\right) = 2\left(\frac{\pi}{2} - \tan^{-1}\left(\sqrt{\frac{b}{a}}\right)\right) = 2\left(\cot^{-1}\left(\sqrt{\frac{b}{a}}\right)\right) = 2\tan^{-1}\left(\sqrt{\frac{a}{b}}\right)$$

Therefore, both the unequal interior angles of C-I trapezium having parallel sides of lengths a and b, are given as follows

$$\boldsymbol{2\theta = 2\tan^{-1}\left(\sqrt{\frac{b}{a}}\right), \quad \pi - 2\theta = 2\tan^{-1}\left(\sqrt{\frac{a}{b}}\right)}$$

2.4.3. Equal non-parallel sides (c)

In right ΔIPA (see the above figure-6)

$$\tan\angle IAP = \tan\theta = \frac{IP}{AP} \Rightarrow AP = IP\cot\theta = r\cot\theta$$

Similarly, in right ΔIPD

$$\tan\angle DIP = \tan\theta = \frac{PD}{IP} \Rightarrow PD = IP\tan\theta = r\tan\theta$$

$$\therefore \quad AP + PD = r\cot\theta + r\tan\theta \Rightarrow \quad AD = r(\tan\theta + \cot\theta)$$

$$\Rightarrow c = \frac{\sqrt{ab}}{2}\left(\sqrt{\frac{b}{a}} + \sqrt{\frac{a}{b}}\right) = \frac{1}{2}(a+b)$$

Therefore, each of the equal non-parallel sides of C-I trapezium having parallel sides of lengths a and b, is given as follows

$$\boldsymbol{c = \frac{1}{2}(a+b) = AM}$$

2.4.4. Perpendicular distance between parallel sides (h)

The perpendicular distance h between the parallel sides of lengths a and b of C-I trapezium, is given as follows

$$\boldsymbol{h} = BT = MN = 2r = 2\frac{\sqrt{ab}}{2} = \boldsymbol{\sqrt{ab} = GM}$$

2.4.5. Circum-radius (R)

In right ΔOMC (as shown in the above figure-6), using Pythagorean theorem as follows

$$OM = \sqrt{(OC)^2 - (MC)^2} = \sqrt{(R)^2 - \left(\frac{b}{2}\right)^2} = \sqrt{R^2 - \frac{b^2}{4}}$$

Similarly, in right ΔONB (see the above figure-6), using Pythagorean theorem as follows

$$ON = \sqrt{(OB)^2 - (NB)^2} = \sqrt{(R)^2 - \left(\frac{a}{2}\right)^2} = \sqrt{R^2 - \frac{a^2}{4}}$$

$$\Rightarrow OM + ON = MN$$

$$\sqrt{R^2 - \frac{b^2}{4}} + \sqrt{R^2 - \frac{a^2}{4}} = \sqrt{ab} \qquad (\because MN = h = \sqrt{ab})$$

$$\sqrt{R^2-\frac{a^2}{4}}=\sqrt{ab}-\sqrt{R^2-\frac{b^2}{4}}$$

$$\left(\sqrt{R^2-\frac{a^2}{4}}\right)^2=\left(\sqrt{ab}-\sqrt{R^2-\frac{b^2}{4}}\right)^2$$

$$R^2-\frac{a^2}{4}=ab+R^2-\frac{b^2}{4}-2\sqrt{ab}\sqrt{R^2-\frac{b^2}{4}}$$

$$\sqrt{R^2-\frac{b^2}{4}}=\frac{4ab+a^2-b^2}{8\sqrt{ab}}$$

$$\left(\sqrt{R^2-\frac{b^2}{4}}\right)^2=\left(\frac{4ab+a^2-b^2}{8\sqrt{ab}}\right)^2$$

$$R^2-\frac{b^2}{4}=\frac{(4ab+a^2-b^2)^2}{64ab}$$

$$R^2=\frac{(4ab+a^2-b^2)^2+16ab^3}{64ab}$$

$$R^2=\frac{16a^2b^2+a^4+b^4+8a^3b-2a^2b^2-8ab^3+16ab^3}{64ab}$$

$$R^2=\frac{a^4+b^4+14a^2b^2+8a^3b+8ab^3}{64ab} \qquad \dots\dots..Eq(3)$$

$$R^2=\frac{(a^4+b^4+18a^2b^2+8a^3b+8ab^3)-4a^2b^2}{64ab}$$

$$R^2=\frac{(a^2+b^2+4ab)^2-(2ab)^2}{64ab}$$

$$R^2=\frac{(a^2+b^2+4ab+2ab)(a^2+b^2+4ab-2ab)}{64ab}$$

$$R^2=\frac{(a^2+b^2+6ab)(a^2+b^2+2ab)}{64ab}=\frac{(a^2+b^2+6ab)(a+b)^2}{64ab}$$

$$\Rightarrow \quad R=\sqrt{\frac{(a^2+b^2+6ab)(a+b)^2}{64ab}}$$

$$R=\frac{a+b}{8}\sqrt{\frac{a^2+b^2+6ab}{ab}}$$

$$R=\frac{a+b}{4}\sqrt{\frac{(a+b)^2+4ab}{4ab}}=\frac{a+b}{4}\sqrt{\frac{(a+b)^2}{4ab}+1}$$

$$R=\frac{1}{2}\cdot\frac{a+b}{2}\sqrt{\left(\frac{a+b}{2\sqrt{ab}}\right)^2+1}=\frac{1}{2}\left(\frac{a+b}{2}\right)\sqrt{\left(\frac{\frac{a+b}{2}}{\sqrt{ab}}\right)^2+1}$$

$$R=\frac{1}{2}(AM)\sqrt{\left(\frac{AM}{GM}\right)^2+1}$$

Where, AM is arithmetic mean and GM is geometric mean of parallel sides a and b of C-I trapezium.

Therefore, the radius R of circumscribed circle of C-I trapezium having parallel sides of lengths a and b, is given as follows

$$\boldsymbol{R=\frac{a+b}{8}\sqrt{\frac{a^2+b^2+6ab}{ab}}=\frac{1}{2}(AM)\sqrt{\left(\frac{AM}{GM}\right)^2+1}}$$

Where, $AM=(a+b)/2$ and $GM=\sqrt{ab}$

2.4.6. Area (Δ_{CIT})

Now, the area Δ_{CIT} of circum-inscribed trapezium with parallel sides a and b which have a normal distance h between them, is given as follow

$$\Delta_{CIT}=\text{Area of trapezium with parallel sides } a\ \&\ b \text{ at a normal distance } h$$

$$\Delta_{CIT}=\frac{1}{2}(a+b)(h)=\frac{1}{2}(a+b)\left(\sqrt{ab}\right)=AM\times GM$$

$$\therefore\ \boldsymbol{\Delta_{CIT}=\frac{1}{2}(a+b)\sqrt{ab}=AM\times GM}$$

2.4.7. Perimeter (P_{CIT})

Now, the perimeter P_{CIT} of circum-inscribed trapezium with parallel sides a and b, is given as follow

$$P_{CIT}\ =\text{Perimeter of trapezium with sides } a, b, c \text{ and } c$$

$$P_{CIT}\ =a+b+c+c=a+b+2c=a+b+2\left(\frac{a+b}{2}\right)=2(a+b)=4(AM)$$

$$\therefore\ \boldsymbol{P_{CIT}\ =2(a+b)=4(AM)}$$

2.4.8. Length of equal diagonals (L_{diag})

Now, using cosine rule in ΔABC (see the above figure-6) as follows

$$\cos\angle ABC=\frac{(AB)^2+(BC)^2-(AC)^2}{2(AB)(BC)}\qquad\Rightarrow\quad AC=\sqrt{(AB)^2+(BC)^2-2(AB)(BC)\cos\angle ABC}$$

$$\Rightarrow\ AC=\sqrt{(a)^2+\left(\frac{a+b}{2}\right)^2-2(a)\left(\frac{a+b}{2}\right)\cos 2\theta}\qquad\text{(Substituting the values)}$$

$$AC=\sqrt{\frac{4a^2+(a+b)^2}{4}-a(a+b)\cdot\frac{1-\tan^2\theta}{1+\tan^2\theta}}$$

$$AC=\sqrt{\frac{4a^2+(a+b)^2}{4}-a(a+b)\cdot\frac{1-\frac{b}{a}}{1+\frac{b}{a}}}\qquad\left(\because\ \tan\theta=\sqrt{\frac{b}{a}}\right)$$

$$AC=\sqrt{\frac{4a^2+(a+b)^2}{4}-a(a+b)\cdot\frac{a-b}{a+b}}=\sqrt{\frac{4a^2+(a+b)^2}{4}-a(a-b)}$$

$$AC=\sqrt{\frac{4a^2+a^2+b^2+2ab-4a^2+4ab}{4}}=\sqrt{\frac{a^2+b^2+6ab}{4}}=\frac{1}{2}\sqrt{a^2+b^2+6ab}$$

$$AC=\sqrt{\frac{(a+b)^2+4ab}{4}}=\sqrt{\frac{(a+b)^2}{4}+ab}=\sqrt{\left(\frac{a+b}{2}\right)^2+\left(\sqrt{ab}\right)^2}$$

$$\Rightarrow\quad AC=\sqrt{(AM)^2+(GM)^2}\quad\Rightarrow\quad BD=AC=\sqrt{(AM)^2+(GM)^2}$$

Therefore, the length L_{diag} of each of two equal diagonals of C-I trapezium having parallel sides of lengths a and b, is given as follows

$$\boldsymbol{L_{diag}=\frac{1}{2}\sqrt{a^2+b^2+6ab}=\sqrt{(AM)^2+(GM)^2}}$$

Where, $AM=(a+b)/2$ and $GM=\sqrt{ab}$

2.4.9. Distance between circum-centre and in-centre (CI)

Now, in right ΔONB (see the above figure-6), using Pythagorean theorem as follows

$$ON=\sqrt{(OB)^2-(NB)^2}=\sqrt{(R)^2-\left(\frac{a}{2}\right)^2}$$

$$ON=\sqrt{\frac{a^4+b^4+14a^2b^2+8a^3b+8ab^3}{64ab}-\frac{a^2}{4}}\qquad\left(\text{Setting value of } R^2 \text{ from Eq(3)}\right)$$

$$ON=\sqrt{\frac{a^4+b^4+14a^2b^2-8a^3b+8ab^3}{64ab}}=\sqrt{\frac{(b^2-a^2+4ab)^2}{64ab}}=\frac{b^2-a^2+4ab}{8\sqrt{ab}}\qquad(\because ON\geq 0)$$

From the above figure-6,

$$\mathrm{OI}=ON-IN=\frac{b^2-a^2+4ab}{8\sqrt{ab}}-\frac{\sqrt{ab}}{2}=\frac{b^2-a^2+4ab-4ab}{8\sqrt{ab}}=\frac{b^2-a^2}{8\sqrt{ab}}$$

$$\mathrm{OI}=\frac{|a^2-b^2|}{8\sqrt{ab}}=\frac{|(a-b)(a+b)|}{8\sqrt{ab}}=\frac{|a-b|(a+b)}{8\sqrt{ab}}=\frac{|a-b|}{4}\frac{\frac{(a+b)}{2}}{\sqrt{ab}}=\frac{|a-b|}{4}\cdot\frac{AM}{GM}$$

Therefore, the distance CI between circum-centre O and in-centre I of C-I trapezium having parallel sides of lengths a and b, is given as follows

$$\mathbf{CI} = \frac{|a^2-b^2|}{8\sqrt{ab}} = \frac{|a-b|}{4}\left(\frac{AM}{GM}\right)$$

Where, $AM = (a+b)/2$ and $GM = \sqrt{ab}$

2.4.10. Angle between the diagonals (δ)

Let δ be the angle between the equal diagonals AC and BD which intersect each other at the point E in C-I trapezium ABCD (as shown in the figure-7). Since the equal diagonal intersect each other in the same ratio hence using symmetry in ΔABE, $AE = BE \Rightarrow \angle BAE = \angle ABE = \alpha$ (Let).

Now, using Cosine Rule in ΔADB (see figure-7) as follows

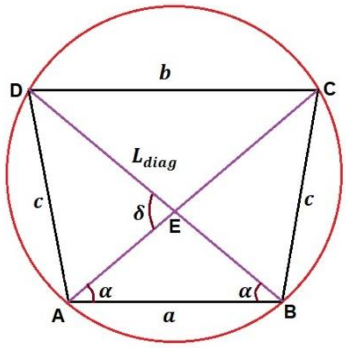

Figure-7: The diagonals AC and BD intersect each other in the same ratio, AE:EC::BE:ED.

$$\cos\angle ABD = \frac{(AB)^2+(BD)^2-(AD)^2}{2(AB)(BD)} = \frac{(a)^2+\left(L_{diag}\right)^2-(c)^2}{2(a)\left(L_{diag}\right)}$$

$$\cos\alpha = \frac{a^2+\left(\frac{1}{2}\sqrt{a^2+b^2+6ab}\right)^2-\left(\frac{a+b}{2}\right)^2}{2(a)\left(\frac{1}{2}\sqrt{a^2+b^2+6ab}\right)}$$

$$\cos\alpha = \frac{4a^2+a^2+b^2+6ab-a^2-b^2-2ab}{4a\sqrt{a^2+b^2+6ab}}$$

$$\cos\alpha = \frac{a+b}{\sqrt{a^2+b^2+6ab}}$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{a+b}{\sqrt{a^2+b^2+6ab}}\right) = \sec^{-1}\left(\frac{\sqrt{a^2+b^2+6ab}}{a+b}\right) = \sec^{-1}\left(\sqrt{\frac{a^2+b^2+6ab}{(a+b)^2}}\right)$$

$$\alpha = \sec^{-1}\sqrt{\frac{(a+b)^2+4ab}{(a+b)^2}} = \sec^{-1}\sqrt{1+\frac{4ab}{(a+b)^2}} = \sec^{-1}\sqrt{1+\left(\frac{2\sqrt{ab}}{a+b}\right)^2} = \sec^{-1}\sqrt{1+\left(\frac{\sqrt{ab}}{\frac{a+b}{2}}\right)^2}$$

$$\alpha = \sec^{-1}\sqrt{1+\left(\frac{GM}{AM}\right)^2} \qquad \left(\text{Where, } AM = \frac{a+b}{2},\ GM = \sqrt{ab}\right)$$

From the above figure-7, the angle δ is the exterior angle of isosceles ΔABE therefore angle δ will be equal to the sum of opposite interior angles $\angle BAE$ and $\angle ABE$ i.e.

$$\delta = \angle BAE + \angle ABE = \alpha + \alpha = 2\alpha = 2\sec^{-1}\sqrt{1+\left(\frac{GM}{AM}\right)^2}$$

Therefore, the supplementary angles of intersection of diagonals of C-I trapezium having parallel sides of lengths a and b, are given as follows

$$\boldsymbol{\delta = 2\sec^{-1}\left(\frac{\sqrt{a^2+b^2+6ab}}{a+b}\right) = 2\sec^{-1}\sqrt{1+\left(\frac{GM}{AM}\right)^2}\ ,\ \pi-\delta = 2\,\mathrm{cosec}^{-1}\sqrt{1+\left(\frac{GM}{AM}\right)^2}}$$

2.4.11. Ratio of intersection of diagonals (R_{diag})

Using Sine Rule in ΔABE (see figure-7 above) as follows

$$\frac{AE}{\sin\angle BAE}=\frac{AB}{\sin\angle AEB}\ \Rightarrow\ \frac{AE}{\sin\alpha}=\frac{a}{\sin(\pi-\delta)}\ \Rightarrow\ \frac{AE}{\sin\alpha}=\frac{a}{\sin(\delta)}\ \Rightarrow\ AE=\frac{a\sin\alpha}{\sin(\delta)}$$

$$\Rightarrow\quad AE=\frac{a\sin\alpha}{\sin(2\alpha)}=\frac{a\sin\alpha}{2\sin\alpha\cos\alpha}=\frac{a}{2}\sec\alpha=\frac{a}{2}\frac{\sqrt{a^2+b^2+6ab}}{a+b}\qquad(\because\ \delta=2\alpha\ )$$

$$\Rightarrow\quad AE=\frac{a\sqrt{a^2+b^2+6ab}}{2(a+b)}\qquad\ldots\ldots\ldots\ldots(4)$$

From the above figure-7, We have

$$EC=AC-AE=\frac{1}{2}\sqrt{a^2+b^2+6ab}-\frac{a\sqrt{a^2+b^2+6ab}}{2(a+b)}=\frac{1}{2}\sqrt{a^2+b^2+6ab}\left(1-\frac{a}{a+b}\right)$$

$$\Rightarrow\ EC=\frac{b\sqrt{a^2+b^2+6ab}}{2(a+b)}\qquad\ldots\ldots\ldots\ldots(5)$$

Since the equal diagonals AC and BD intersect each other at the point E, therefore the ratio of intersection can be determined as

$$\frac{AE}{EC}=\frac{\frac{a\sqrt{a^2+b^2+6ab}}{2(a+b)}}{\frac{b\sqrt{a^2+b^2+6ab}}{2(a+b)}}=\frac{a}{b}\qquad\left(\text{Setting values of } AE\ \&EC \text{ from Eq}(4)\ \&Eq(5)\right)$$

Therefore, the ratio of intersection of diagonals of C-I trapezium having parallel sides of lengths a and b, is given as follows

$$\boldsymbol{AE}:\boldsymbol{EC}::\boldsymbol{a}:\boldsymbol{b}\quad \text{Or}\quad \boldsymbol{BE}:\boldsymbol{ED}::\boldsymbol{a}:\boldsymbol{b}$$

It is worth noticing that all the formula in term of known parallel sides a and b of C-I trapezium have symmetry which implies that interchange of the parallel sides a and b does not change the corresponding geometric parameters.

2.5 Methods of constructing C-I trapezium

Consider a circum-inscribed (C-I) trapezium with its parallel sides a and b. Now, the required C-I trapezium can easily be constructed by following three methods

Method-1: Using circumscribed circle

The following steps can be used to construct a C-I trapezium by using circumscribed circle

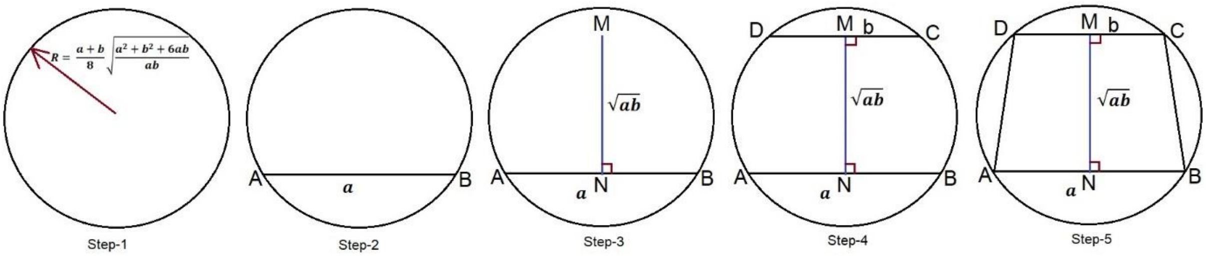

Figure-8: The steps to construct, using circumscribed circle, a C-I trapezium with parallel sides a, and b.

Step-1: Draw the circumscribed circle of radius $R = \frac{a+b}{8}\sqrt{\frac{a^2+b^2+6ab}{ab}}$ (as shown in the figure-7)

Step-2: Draw the chord AB of length a

Step-3: Draw the perpendicular bisector MN of chord AB such that $MN = \sqrt{ab}$

Step-4: Draw the chord CD passing through the point M and perpendicular to MN or parallel to chord AB.

Step-5: Join the end points A & B to the points D & C respectively to obtain the required C-I trapezium ABCD (as shown in the above figure-8).

Method-2: Using inscribed circle

The following steps can be used to construct a C-I trapezium by using inscribed circle

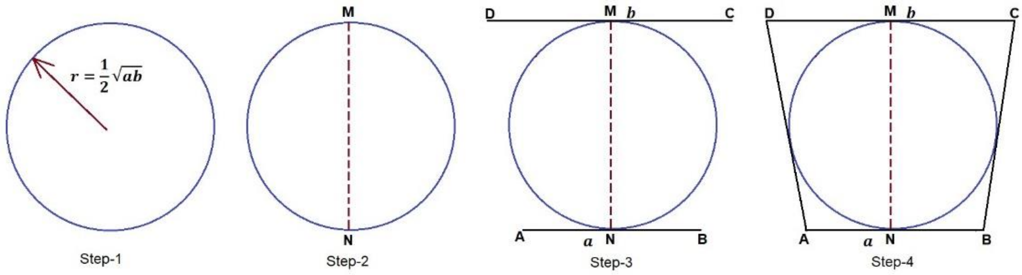

Figure-9: The steps to construct, using inscribed circle, a C-I trapezium with parallel sides a, and b.

Step-1: Draw the inscribed circle of radius $r = \frac{1}{2}\sqrt{ab}$ (as shown in the figure-9)

Step-2: Draw the diameter MN.

Step-3: Draw the tangents AB and CD of length a and b which are bisected at the points of tangency M and N respectively.

Step-4: Join the end points A & B to the points D & C respectively to obtain the required C-I trapezium ABCD (as shown in the above figure-9).

Method-3: Using interior angles

The following steps can be used to construct a C-I trapezium by using interior angles

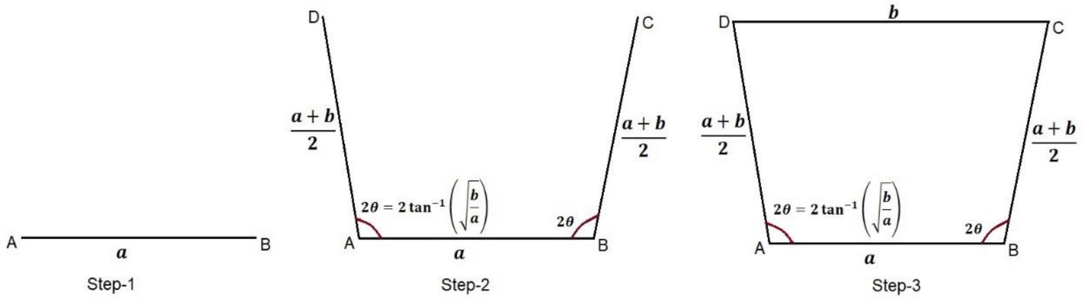

Figure-10: The steps to construct, using interior angles, a C-I trapezium with parallel sides a, and b.

Step-1: Draw the line AB of length a (as shown in the figure-10)

Step-3: Draw two lines AD and BC each of length $\frac{a+b}{2}$ and at an equal angle of $2\theta = 2\tan^{-1}\left(\sqrt{\frac{b}{a}}\right)$ with the line AB (as shown in the above figure-10).

Step-3: Join the end points A & B to the points D & C respectively to obtain the required C-I trapezium ABCD

Summary: If a, and b are two parallel sides of a circum-inscribed trapezium then all of its important geometric parameters can be determined as tabulated below.

Geometric parameter	**Formula**
Each of equal non-parallel sides	$\frac{(a+b)}{2} = AM$
Perpendicular distance between parallel sides	$\sqrt{ab} = GM$
Interior angles (supplementary)	$2\tan^{-1}\left(\sqrt{\frac{b}{a}}\right), \ \ 2\tan^{-1}\left(\sqrt{\frac{a}{b}}\right)$
Each of equal diagonals	$\frac{1}{2}\sqrt{a^2+b^2+6ab} = \sqrt{(AM)^2+(GM)^2}$
Angles between diagonals	$2\sec^{-1}\sqrt{1+\left(\frac{GM}{AM}\right)^2}, 2\,\mathrm{cosec}^{-1}\sqrt{1+\left(\frac{GM}{AM}\right)^2}$
Ratio of intersection of diagonals	$a:b$ Or $b:a$
Radius of circumscribed circle	$\frac{a+b}{8}\sqrt{\frac{a^2+b^2+6ab}{ab}} = \frac{1}{2}(AM)\sqrt{\left(\frac{AM}{GM}\right)^2+1}$

Radius of inscribed circle	$\frac{\sqrt{ab}}{2}=\frac{GM}{2}$
Distance between circum-centre and in-centre	$\frac{\lvert a^2-b^2\rvert}{8\sqrt{ab}}=\frac{\lvert a-b\rvert}{4}\left(\frac{AM}{GM}\right)$
Perimeter	$2(a+b)=4(AM)$
Area	$\frac{1}{2}(a+b)\sqrt{ab}=AM\times GM$

Where, $AM=(a+b)/2$ is Arithmetic Mean and $GM=\sqrt{ab}$ is Geometric Mean of parallel sides a & b of circum-inscribed trapezium.

It is interesting to note that a circum-inscribed trapezium having parallel sides equal in length becomes a square i.e. substituting $AM=(a+a)/2=a$ and $GM=\sqrt{a\cdot a}=a$ in the above formula, the obtained results show that the circum-inscribed trapezium with equal parallel sides is a square.

Conclusions

In this paper, the new term 'circum-inscribed' has been introduced and the parameters of four well known polygons which are circum-inscribed have been explained and analysed in details. The different parameters of a C-I trapezium have been computed in terms of AM and GM of known sides such as radii of circumscribed & inscribed circles, unknown sides, interior angles, diagonals, angles between diagonals, ratio of intersection of diagonals, perimeter, and area of circum-inscribed trapezium. The analytic formula have been derived and established for solving the various problems of 2D and 3D geometrical figures such as cyclic-quadrilateral and polyhedrons.